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Hamiltonian approach to $Z(N)$ lattice gauge theories

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We develop a Hamiltonian formalism for $Z(N)$ lattice gauge theories. Duality is expressed by algebraic operator relations which are the analog of the interchange of electric and magnetic fields in $D = 3$ space dimensions. In $D = 2$ duality is used to solve the gauge condition. This leads to a generalized Ising Hamiltonian. In $D = 3$ our theory is self-dual. For $N \to \infty$ the theory turns into "periodic QED" in appropriate limits. This leads us to propose the existence of three phases for $N > N_c \simeq 6$. Their physical properties can be classified as electric-confining, nonconfining, and magnetic-confining.

I. INTRODUCTION

$Z(N)$ gauge theories have been the subject of intensive investigations during the last year. These theories are interesting on several accounts. The foremost reason is their possible relevance to the question of confinement. 't Hooft[1] and Polyakov[2] argued that the center $Z(3)$ of the SU(3) color gauge group can play a crucial role in the confinement of quarks (or any object with nonzero color triality). Regardless of whether this is true, the $Z(N)$ models are interesting theoretical laboratories because they exhibit confinement for large enough couplings. Moreover, these models provide a realization of 't Hooft's algebra of order and disorder operators which characterize the different possible phases of the SU(N) gauge theories.[1,3]

One way in which $Z(N)$ gauge theory can emerge is a limiting case of an SU(N) lattice gauge theory with Higgs fields which belong to an SU(N)/$Z(N)$ representation; this is the limit in which all Higgs modes together with all gauge degrees of freedom apart from those which belong to the center $Z(N)$ are frozen.[4]

A different interesting aspect of the $Z(N)$ gauge theories on the lattice is that they can be formally related to "periodic QED" (PQED) in the $N \to \infty$ limit. Periodic (or compact) QED is a theory which is of interest by itself,[5] because in $3+1$ dimensions it is known to have a confining as well as a nonconfining phase. We will use the relation between the two models to learn about the phase diagram of the $Z(N)$ theories. Finally, there is a close connection between the $Z(N)$ gauge theories and spin-glass models[6] which are of considerable interest in statistical mechanics.

In Sec. II we present for the sake of completeness a short review of the connection between the $Z(N)$ gauge theories and the confinement problem. We discuss the 't Hooft algebraic approach and its realization. We demonstrate the emergence of $Z(N)$ configurations as solutions to the lattice SU(N) equations of motion which was recently noted by Yoneya.[7] We compare the Hamiltonian approach which we use throughout the paper with the Euclidean action formulation. Section III is devoted to our Hamiltonian formulation of the $Z(N)$ gauge theories with particular emphasis on their gauge invariance. In Sec. IV we discuss the duality transformation in two space dimensions and its role in solving the gauge constraints. The connection with the Ising model and spin-glass problems is explained. The nature of the phases of the two-dimensional $Z(N)$ systems is discussed in Sec. V. The analogous discussions of the three-dimensional models are presented in Secs. VI and VII. In analyzing the phase diagram we make use of the connection between PQED and the large-N limit of the $Z(N)$ theories. Arguments based on this connection as well as on a renormalization-group approach lead to the conclusion that for $N > N_c \simeq 6$ these theories possess three phases. These phases can be characterized as electric-confining, nonconfining, and magnetic-confining.

An interesting question that arises from our work concerns the interrelation between asymptotic freedom and confinement. The common belief is that the SU(N) gauge theories, which are asymptotically free, have only one confining phase. On the other hand SU(N) has as a subgroup its center $Z(N)$. 't Hooft has argued that the large-distance properties of SU(N) are dominated by its $Z(N)$ part. Our work reveals that for large

enough N the $Z(N)$ gauge theories have three phases and the critical point of the electric-confining phase grows as N^2. This, then raises the question of whether the SU(N) gauge theory may have more than one phase for large enough N.

II. $Z(N)$ GAUGE THEORIES AND CONFINEMENT

It is generally believed that quantum chromodynamics (QCD) is the theory underlying the structure of hadrons. The short-distance behavior of QCD can be analyzed perturbatively because this theory is asymptotically free.[8] However, the large-distance structure of QCD is much more difficult to handle. To understand the spectrum of hadrons and quark confinement we have to develop nonperturbative methods. Much progress has been made during recent years towards understanding the infrared structure of the theory. The methods that have been developed can be classified as belonging to either a lattice approach or a semiclassical approach to QCD.

In the lattice approach one regards QCD as the continuum limit of certain field theories formulated on lattices.[9] The lattice field theories are well defined and strictly conserve local gauge invariance.[10] It is a common belief that the latter is essential to achieve confinement. On a large lattice the theory is in a strong-coupling regime and exhibits confinement. Here one calculates the spectrum and then continues to the weak-coupling limit by using the Padé approximation to the strong-coupling expansion[11] or renormalization-group[12] and variational[13] techniques. The success of this approach depends critically on the absence of any phase transition which will prevent the continuation to weak coupling where the continuum field theory is recovered.

Another approach to the problem is to start with the classical QCD action and to try to identify the most important field configurations which saturate the functional integral. Once these are singled out one can check their stability by performing small oscillations around these configurations and one can calculate their contributions to physically significant quantities. This investigation is usually carried out within the gluon sector in which the quarks are regarded as external classical charges.[14] This approach can of course also be used in the study of field theories on the lattice. It was indeed within this context that Wilson introduced the order parameter[9]

$$\langle \mathcal{C}(C)\rangle = \left\langle \mathrm{Tr}\, P \exp\left(i \oint_C A_\mu dx^\mu\right)\right\rangle \equiv \int DA\, e^{-S(A)} \exp\left(i \oint_C A_\mu dx^\mu\right). \tag{2.1}$$

An area dependence of the Wilson loop

$$\langle \mathcal{C}(C)\rangle \sim e^{-\text{area}\,(C)} \tag{2.2}$$

is generally accepted as the criterion for confinement within the Euclidean formulation of the gluon sector of QCD.

't Hooft[1] has recently suggested that field configurations with nontrivial $Z(3)$ topological charges are important in bringing about confinement. He introduced a disorder loop operator $\mathcal{B}(C)$ [in 2+1 dimensions it is a local operator $\phi(x)$] which together with the Wilson loop order operator $\mathcal{C}(C)$ satisfies a $Z(N)$ algebra:

$$\mathcal{C}^\dagger(C')\mathcal{B}(C)\mathcal{C}(C') = \mathcal{B}(C)e^{i2\pi n/N}, \tag{2.3}$$

where n is the number of times the curve C winds around the curve C'. In two space dimensions the proposed commutation relations are

$$\mathcal{C}^\dagger(C)\phi(x)\mathcal{C}(C) = \phi(x)e^{i2\pi n/N}, \tag{2.4}$$

where now n is the number of times the curve C winds around the point x. The operator $\mathcal{B}(C)$ is associated with the topological excitations of the system in much the same way that the operator $\mathcal{C}(C)$ is associated with color-electric excitations. An investigation of the algebra of Eq. (2.3) enabled 't Hooft to characterize the various phases of the system. In particular, the confining phase is the disordered one where in addition to Eq. (2.2) he finds

$$\begin{aligned} &\langle \mathcal{B}(C)\rangle \sim e^{-\text{length}\,(C)} && \text{in } 3+1 \text{ dimensions},\\ &\langle \phi(x)\rangle \neq 0 && \text{in } 2+1 \text{ dimensions}. \end{aligned} \tag{2.5}$$

In this phase the $Z(N)$ topological configurations are dominant.

This confinement is a confinement of triality [the center of SU(3) of color]. A linear asymptotic potential exists in this phase between triality-nonzero sources. Although this is sufficient to account for the nonobservation of quarks, the $Z(3)$ model cannot describe correctly the binding force for low-mass hadrons: If the binding force had been only triality dependent it would have led to degenerate mesons in which the $q\bar{q}$ pair can be either a singlet or an octet of color. This degeneracy would be lifted by the short-distance color-electric component of the QCD force and result in a much richer spectrum than the one observed. Hence one should regard this $Z(3)$ approach as being a candidate only for the large-distance behavior of the confining mechanism.

The $Z(N)$ topological configurations are singular in the absence of Higgs fields. Since we are anyway interested in their effects on the large-distance component of the force we may study them in lattice QCD theories. Moreover, the $Z(N)$

gauge theories which provide the realization of the 't Hooft algebra do not have a continuum limit since there are no infinitesimal $Z(N)$ transformations. Let us present now an argument for the emergence of the $Z(N)$ components of the QCD lattice theory in accordance with the observation of Yoneya.[7] We start with Wilson's action which can be written as

$$S=\frac{1}{g}\sum_{p}\mathrm{Tr}(U_{p1}^{\dagger}U_{p2}^{\dagger}U_{p3}U_{p4}+\mathrm{H.c.}-2)$$
$$\equiv\frac{1}{g}\sum_{p}\mathrm{Tr}(X_p+\mathrm{H.c.}-2)\,, \tag{2.6}$$

where the U_l are defined on the links of the lattice and are assumed to be elements of the SU(N) group. The summation in Eq. (2.6) is carried out over all the plaquettes of the lattice and the notation is explained in Fig. 1. The equations of motion of this Euclidean field theory are obtained by varying independently on each link both U_l and $U_l^{\dagger}$ subject to the constraint $U_l^{\dagger}U_l=1$. Self-consistently we can diagonalize each $N\times N$ matrix X_p,

$$(X_p)_{ij}=a_{p,i}\delta_{ij}\,, \tag{2.7}$$

and minimize $\mathrm{Tr}X_p$ subject to the constraint

$$\det X_p=\prod_{i=1}^{N}a_{p,i}=1 \tag{2.8}$$

by introducing the Lagrange multipliers λ_p:

$$f_p=\mathrm{Tr}X_p-\lambda_p\prod_{i=1}^{N}a_{p,i}\,,$$
$$0=\frac{\partial f_p}{\partial a_{p,i}}=1-\lambda_p\prod_{j=1}^{i-1}\prod_{j=i+1}^{N}a_{p,j}\,. \tag{2.9}$$

Using the constraint (2.8) we find $a_{p,i}=\lambda_p$ and

$$a_{p,i}=e^{i2\pi n_p/N}\quad(n_p=\text{integer})\ \text{or}\ X_p\in Z(N)\,. \tag{2.10}$$

If we restrict ourselves to the subgroup $U_l\in Z(N)$ then Eq. (2.10) is satisfied, i.e., we deal with group elements which are extrema of the action (2.6). As long as only a finite number of links have $U\neq 1$, the resulting configuration has a finite action. Yoneya[7] carried out a detailed analysis of the stability of such $Z(N)$ configurations for $N=2,3,4$.

At this point we have the option of using either an action or a Hamiltonian formulation of the $Z(N)$ lattice field theory. There exists a well-defined way of constructing the Hamiltonian out of the Euclidean action formulation.[3] The idea is to choose one of the Euclidean axes as the time axis and construct a transfer matrix which is identified as $e^{-H\Delta t}$. Taking the lattice spacing in this direction to zero and rescaling the appropriate coupling constants accordingly leads to the construction of the Hamiltonian in terms of quantum operators defined on a lattice in space. The action formulation is like the partition function formalism in statistical mechanics and is defined over classical variables. Thus all the $Z(N)$ elements in the action formulation of the lattice field theory commute with one another. Going from the action formulation to the Hamiltonian formulation the dimension gets diminished by one unit; however, the number of variables per link is doubled and they obey nontrivial commutation relations. Rather than starting with the action formulation we will work directly with the Hamiltonian quantum-mechanical formalism.

The phase diagrams of the $Z(N)$ gauge theories will be one of our main interests. A well-known tool for analyzing the location of their phase-transition points is the duality transformation.[15] The $Z(N)$ gauge theories which result from Wilson's action (2.6) in the way discussed above are known to be self-dual only for $N=2,3,4$.[4,16] In Appendix A we present a simple explanation for why that is so. Using our Hamiltonian formulation we will construct $Z(N)$ gauge theories which are self-dual for every N in three space dimensions. For $N>5$ these theories deviate from the models defined through Wilson's action. Our models have, though, simple large-N properties which allow us to make the connection with PQED in the $N\to\infty$ limit.

III. THE $Z(N)$ MODEL AND ITS GAUGE INVARIANCE

Our $Z(N)$ models are defined by a one-parameter (λ) family of Hamiltonians

$$H=-\tfrac{1}{2}\lambda\sum_{l}(P_l^{\dagger}+P_l-2)$$
$$-\sum_{p}\tfrac{1}{2}(Q_{p1}^{\dagger}Q_{p2}^{\dagger}Q_{p3}Q_{p4}+\mathrm{H.c.}-2)\,. \tag{3.1}$$

The unitary operators P_l and Q_l are associated with the links l of the lattice and obey the $Z(N)$ algebra:

$$P_l^N=Q_l^N=1\,,\quad P_l^{\dagger}P_l=Q_l^{\dagger}Q_l=1\,, \tag{3.2}$$
$$P_l^{\dagger}Q_lP_l=e^{i\delta}Q_l\,,\quad\delta=2\pi/N\,. \tag{3.3}$$

Operators which are associated with different links commute with one another. The second term in Eq. (3.1) involves a sum over all plaquettes of the lattice. For each plaquette p one defines the links $p1$ and $p4$ to be parallel to the unit axes of the lattice. The order 1 to 4 defines a closed loop around the plaquette as shown in Fig. 1.

The Hamiltonian (3.1) is locally gauge invariant,

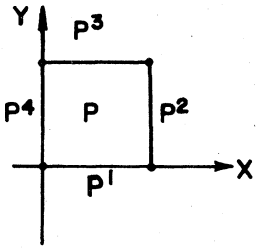

FIG. 1. Notation of links around a plaquette.

i.e., with each lattice site i one can associate a unitary operator $G(i)$ which commutes with the Hamiltonian

$$[G(i), H] = 0 \tag{3.4}$$

and has therefore to be simultaneously diagonalized. $G(i)$ is given by

$$G(i) = \prod_{l_+ \ni i} P^{\dagger}_{l_+} \prod_{l_- \ni i} P_{l_-}, \tag{3.5}$$

where the various links l which touch the vertex i are defined to be positive or negative according to whether they are parallel or antiparallel to the basis vectors (see Fig. 2).

The Hilbert space is divided into sectors which are classified by the eigenvalues of all $G(i)$. One special sector is the gauge-invariant one,

$$G(i)|\psi\rangle = |\psi\rangle \quad \text{for all } i. \tag{3.6}$$

Other sectors, where at some point i one has $G(i) = e^{-i\delta n}$ (n = integer), may be interpreted as representing the physics involving a charge n (mod N) located at that point i. This interpretation is warranted by the following representations of the operators P_l and Q_l:

$$P_l = e^{i\delta E_l}, \quad Q_l = e^{iA_l}. \tag{3.7}$$

E_l and A_l are dimensionless Hermitian operators associated with the link l which are the analogs of the electric field and vector potential. To justify this interpretation we have to demonstrate that E and A obey the commutation relation

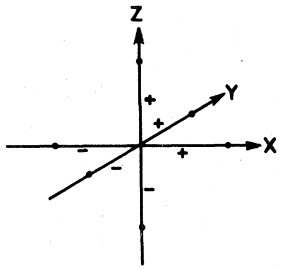

FIG. 2. Classification of positive and negative links around a point.

$$[E_l, A_m] = i\delta_{l,m}. \tag{3.8}$$

Although formally Eq. (3.8) guarantees that (3.7) is a solution to the $Z(N)$ algebra, it cannot strictly hold for finite N because the commutation relation (3.8) cannot be represented on a finite norm set of states. An example of an explicit representation of the $Z(N)$ algebra is given by

$$E|n\rangle = n|n\rangle, \quad n = 0, 1, \ldots, N-1 \tag{3.9}$$

$$e^{iA}|n\rangle = |n+1\rangle \quad \text{for } n = 0, 1, \ldots, N-2, \qquad e^{iA}|N-1\rangle = |0\rangle. \tag{3.10}$$

If it were not for the last equation (3.10) E and A would have obeyed

$$[E, e^{\pm iA}] = \pm e^{\pm iA}. \tag{3.11}$$

The cyclic behavior of Q stands in the way of Eq. (3.11). For large enough N and for appropriate test functions this limitation can be discarded and Eq. (3.11) as well as (3.8) can be retrieved. This problem was discussed by Schwinger.[17] We present a brief discussion of the E and A algebra in Appendix B.

Inserting Eq. (3.7) in the definition of G in Eq. (3.5) we find

$$G(i) = e^{-i\delta \nabla \cdot E}, \tag{3.12}$$

where $\nabla \cdot E$ is the obvious lattice definition of the divergence of the electric field

$$\nabla \cdot E(i) = \sum_{l_+ \ni i} E_{l_+} - \sum_{l_- \ni i} E_{l_-}. \tag{3.13}$$

The above-mentioned identification of the different sectors of Hilbert space with fixed charges at vertices where $G(i) \neq 1$ is the implementation of the Gauss law

$$\nabla \cdot E = \rho. \tag{3.14}$$

In this paper we will not consider the more general problem where the charges are carried by quantum degrees of freedom. The various sectors of Hilbert space will therefore remain separated by superselection rules.

The Hamiltonian (3.1) can now be expressed in terms of electromagnetic-type variables only. We introduce the magnetic field B_p as a plaquette variable using the obvious lattice definition of curl A:

$$B_p = (\nabla \times A)_p = A_{p1} + A_{p2} - A_{p3} - A_{p4}. \tag{3.15}$$

This allows us to rewrite (3.1) in the form

$$H = \lambda \sum_l (1 - \cos\delta E_l) + \sum_p (1 - \cos B_p). \tag{3.16}$$

A link on which $E_l \neq 0$ will be said to carry electric flux and a plaquette on which $B_p \neq 0$ will be

interpreted as carrying magnetic flux. This nomenclature is consistent with that of the QCD theory from which the $Z(N)$ formalism can be derived. These fluxes are defined modulo N, thus for $N=3$, three unit electric flux lines can meet at one point of the lattice. This is, of course, a basic property of a string picture of the structure of baryons. The electric flux lines can be interpreted as strings connecting quarks. We will study the tension of such strings. The representation of Eq. (3.16) is also the appropriate one needed to demonstrate the connection to PQED in the $N\to\infty$ limit.

IV. THE DUALITY TRANSFORMATION IN TWO SPACE DIMENSIONS

We start our discussion of the model in $D=2$ space dimensions by restricting ourselves to the gauge-invariant sector of the theory where $G(i)=1$. Within each sector the variables of the theory are interdependent. We propose to switch to a new set of variables which are independent, i.e., to solve Eq. (3.6) in an operator language. This is achieved by representing each link operator P_l as the product of two unitary operators which are associated with the two plaquettes touching that link:

$$P_l = S^\dagger_{p'} S_{p''} (\delta_{l,p'1}\delta_{l,p''3} + \delta_{l,p'2}\delta_{l,p''4}) . \tag{4.1}$$

The notation is that of Fig. 1. If the operators S obey

$$S_p^\dagger S_p = 1 , \quad S_p^N = 1 , \tag{4.2}$$

then Eq. (3.6) is automatically satisfied. Let us introduce another set of plaquette operators R_p defined by

$$R_p = Q_{p1} Q_{p2} Q^\dagger_{p3} Q^\dagger_{p4} . \tag{4.3}$$

These are evidently also unitary and the Nth root of the identity. In order that the P_l and Q_l obey the $Z(N)$ algebra of Eq. (3.3), also R_p and S_p have to obey the same kind of commutation relations:

$$R_p^\dagger S_p R_p = e^{i\delta} S_p , \quad [R_p, S_q] = 0 \quad \text{if } p \neq q . \tag{4.4}$$

The consistency of (4.4) and (3.3) can be easily checked. One can also see it from the following explicit construction of S_p in terms of a chain of P_l operators which is a solution to Eq. (4.1): With each point of the two-dimensional lattice (i,j), one associates two link operators $P_x(i,j)$ and $P_y(i,j)$ as well as one plaquette operator $S(i,j)$ as shown in Fig. 3. The following substitution

$$S(i,j) = \prod_{j'=0}^{j} P_x^\dagger (i,j') , \tag{4.5}$$

which involves a chain of $P_x^\dagger$ operators extending

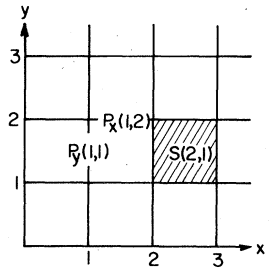

FIG. 3. Notation of link and plaquette variables on a two-dimensional lattice.

to the boundary of the lattice, is a realization of the equation

$$P_x(i,j) = S^\dagger(i,j) S(i,j-1) , \tag{4.6}$$

which is contained in (4.1). The other part of that set of equations, i.e.,

$$P_y(i,j) = S^\dagger(i-1,j) S(i,j) , \tag{4.7}$$

is also satisfied, since in the gauge-invariant sector one can express the P_y operator in terms of chains of P_x operators:

$$P_y(i,j) = \prod_{j'=0}^{j} P_x(i-1,j') P_x^\dagger(i,j') . \tag{4.8}$$

Thus one could choose the P_x operators to be the independent ones and the P_y operators to be the dependent ones. The representation in terms of S_p operators does not necessitate such a gauge choice.

Since the centers of the plaquettes may be regarded as the vertex points of the dual lattice, one refers to the transformation from the set of operators P_l and Q_l to the set R_p and S_p as a dual transformation. The number of operators gets reduced by a factor of 2 since the number of plaquettes is half the number of links in a two-dimensional lattice. The Hamiltonian can be written in terms of the dual variables as

$$H_D = -\tfrac{1}{2}\lambda \sum_{\langle pq \rangle} (S_p^\dagger S_q + S_q^\dagger S_p - 2) - \tfrac{1}{2} \sum_p (R_p + R_p - 2) , \tag{4.9}$$

where the symbol $\langle pq \rangle$ denotes nearest neighbors. Equation (4.9) is a generalized version of an Ising model in a transverse field.[18]

H_D is a correct representation of the volume part of the original Hamiltonian. However, using the explicit solution (4.5) one finds that there are also boundary terms

$$H \doteq H_D - \tfrac{1}{2}\lambda \sum_{\text{bound}} (S_p^\dagger + S_p - 2) . \tag{4.10}$$

The sum includes all plaquettes which lie on the boundary counting twice the four corner ones. The important physical effect of the boundary term is that it lifts the N-fold degeneracy of the vacuum in the high-λ region.

The dot over the equality sign in Eq. (4.10) is there to remind us that this equality holds only in the gauge-invariant sector. If there are points where $G(i) \neq 1$, one cannot use Eq. (4.1). To account for all the sectors using a unified dual notation one has to replace (4.1) with

$$P_l = J_l S^\dagger_{p'} S_{p''} (\delta_{l,p'1}\delta_{l,p''3} + \delta_{l,p'2}\delta_{l,p''4}) , \qquad (4.11)$$

where J_l is a classical variable which can be assigned the values $e^{in\delta}$ with integer n. The gauge operator becomes

$$G(i) = \prod_{l_+ \ni i} J^\dagger_{l_+} \prod_{l_- \ni i} J_{l_-} . \qquad (4.12)$$

Thus the set $\{J_l\}$ determines the sector of Hilbert space to which the solution (4.11) applies. There are, however, many $\{J_l\}$ sets which describe the same sector. Suppose, for example, that the sector has two opposite unit charges:

$$G(i) = \begin{cases} e^{i\delta} \text{ at } i = b , \\ e^{-i\delta} \text{ at } i = a , \\ 1 \text{ otherwise} . \end{cases} \qquad (4.13)$$

Let us define a path Γ on the lattice connecting the points a and b. If we assign $J_l = 1$ to all links which do not lie on Γ the other J are determined as in Fig. 4. Obviously there has to be electric flux flowing between the points a and b. The path of its flow will be determined by the dynamics of the problem, and there is some probability that it will coincide with the path Γ. We can always view Γ as a path along which the electric flux is returned in an equivalent configuration defined in the gauge-invariant sector. This Dirac-string interpretation is depicted in Fig. 5.

Using the new description of P_l we rewrite the

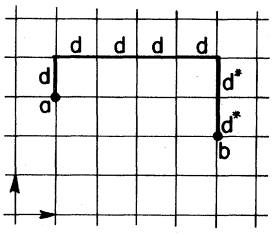

FIG. 4. Choice of J variables in the sector defined by Eq. (4.13) with a unit charge at a and the opposite charge at b. Here $d = e^{i\delta}$.

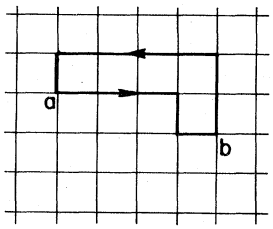

FIG. 5. An electric flux tube in the gauge-invariant sector which has the flux returning along the same path shown in Fig. 4. Regarding this path as a Dirac string the rest may be viewed as the electric field due to charges located at a and b.

dual Hamiltonian as

$$H_D = -\tfrac{1}{2}\lambda \sum_{\langle pq \rangle} (J_{pq} S^\dagger_p S_q + \text{H.c.} - 2) - \tfrac{1}{2} \sum_p (R^\dagger_p + R_p - 2) . \qquad (4.14)$$

This is a generalization of a spin-glass Ising model.[6] In the terminology of statistical mechanics one refers to a situation in which $G(i) \neq 1$ as representing a frustration.[19,20] H_D of Eq. (4.14) is invariant under the local gauge transformation

$$S_q \to e^{i\delta} S_q , \quad J_{pq} \to e^{-i\delta} J_{pq} , \quad \text{any } q, \text{ all } p . \qquad (4.15)$$

This is yet another way of representing the gauge invariance of the original Hamiltonian (3.1). Equation (4.9), which represents only one sector of the theory, is invariant only under the global transformation

$$S_q \to e^{i\delta} S_q , \quad \text{all } q \qquad (4.16)$$

which gets broken by the boundary term in Eq. (4.10).

Let us turn now to the physical interpretation of the operators R and S. Using the electromagnetic representation of Sec. III we see that R_p is the Wilson loop operator defined over one plaquette:

$$R_p = e^{i(A_{p1} + A_{p2} - A_{p3} - A_{p4})} . \qquad (4.17)$$

The Wilson loop operator for any curve C on the lattice is then defined by

$$\mathcal{C}(C) = \exp\left(i \oint_C \vec{A} \cdot d\vec{l} \right) = \prod_{p \subset C} R_p . \qquad (4.18)$$

Using the algebra of R and S we find

$$\mathcal{C}^\dagger(C) S_p \mathcal{C}(C) = e^{im\delta} S_p , \qquad (4.19)$$

where $m = 0$ if p lies outside the curve C, $m = 1$ if p lies inside the curve, and C is topologically equivalent to a circle. This can be generalized to more complicated curves with m representing the number of times the loop C winds around the

plaquette p. S_p plays the role of the monopole field $\phi(x)$ in 't Hooft's topological $Z(N)$ algebra. The monopole interpretation can be understood in the following way: The Wilson loop operator is

$$\mathfrak{a}(C) = \exp\left(i \int_C \int d\sigma B_z\right), \tag{4.20}$$

and Eq. (4.19) implies that

$$\langle\psi|S_p^\dagger \mathfrak{a}(C) S_p|\psi\rangle = \langle\psi|\mathfrak{a}(C)|\psi\rangle \begin{cases} e^{-i\delta} & \text{if } p \subset C, \\ 1 & \text{if } p \not\subset C. \end{cases} \tag{4.21}$$

Hence $S_p^\dagger$ is an operator which creates one unit ($\delta = 2\pi/N$) of magnetic flux flowing through the plaquette p.

To summarize this section let us recall the important aspects of the duality transformations in two space dimensions. These transformations led, within the gauge-invariant sector, to a new set of variables whose number is half that of the original variables. The new operators are independent, and the new Hamiltonian is a generalization of an Ising model. All other sectors can be recovered by changing to a spin-glass formalism. Using the dual variables we reconstruct the topological operators of 't Hooft.

V. THE PHASES OF THE $Z(N)$ THEORY IN TWO SPACE DIMENSIONS

Perturbative treatment of the Hamiltonian (3.1) in the weak- and strong-coupling regimes reveals two different phases. As $\lambda \to \infty$ the vacuum approaches the state characterized by

$$P_l|0\rangle = |0\rangle, \quad S_p|0\rangle = |0\rangle \text{ for every } l \text{ and } p \text{ as } \lambda \to \infty. \tag{5.1}$$

The second part of this equation follows from the first part due to the explicit representation of Eq. (4.5). Alternatively one can see the choice of $S_p = 1$ as following from the boundary terms in Eq. (4.10). The Hamiltonian H_D of Eq. (4.9) which does not have this boundary term has a degenerate vacuum structure of an Ising model.

The other extreme limit $\lambda \to 0$ has a vacuum characterized by

$$R_p|0\rangle = |0\rangle \text{ for every } p \text{ at } \lambda = 0, \tag{5.2}$$

as is easily seen from Eq. (4.9) of H_D. Q_l does not acquire a vacuum expectation value because it is not a gauge-invariant operator.[10]

Alternatively one can consider the Wilson loop (4.18) which has the following behavior in the two limits:

$$\langle\mathfrak{a}(C)\rangle_0 \to \begin{cases} 1 \text{ for } \lambda \to 0 \\ \\ 0 \text{ for } \lambda \to \infty \end{cases} \text{ for every } C. \tag{5.3}$$

The vacuum expectation values of $\mathfrak{a}(C)$ and S_p have opposite trends in the two extreme limits of λ and can be interpreted as order and disorder parameters, respectively.[1,3] Moreover, in the large-coupling regime, the first nonzero contribution to $\langle\mathfrak{a}(C)\rangle_0$ is obtained in that order of perturbation theory which is equal to the area of the loop C; hence,

$$\langle\mathfrak{a}(C)\rangle_0 \sim \left(\frac{1}{\lambda}\right)^{\text{area}(C)} = e^{-\text{area}(C)\ln\lambda} \quad \text{for } \lambda \gg 1. \tag{5.4}$$

The high-λ regime obeys the Wilson criterion for confinement, Eq. (2.2). For small λ a perturbative calculation reveals a perimeter law behavior:

$$\langle\mathfrak{a}(C)\rangle_0 \sim e^{-c\lambda^2 \text{perimeter}(C)}, \quad \text{for } \lambda \ll 1. \tag{5.5}$$

Such arguments for the $Z(2)$ theory were presented by Fradkin and Susskind.[3] The Hamiltonian approach permits the discussion of the different phases in terms of more direct physical quantities: the electric string tension[3] in the high-λ range and the mass of the magnetic monopole in the low-λ range.

To define the tension of a string of electric flux we study the sector of Hilbert space defined by

$$G(i) = \begin{cases} e^{i\delta} \text{ at } x = \frac{1}{2}L, \quad y = 0 \\ e^{-i\delta} \text{ at } x = -\frac{1}{2}L, \quad y = 0 \\ 1 \text{ otherwise}. \end{cases} \tag{5.6}$$

L is supposed to be very large compared to the (unit) size of the lattice link. The string state is defined to be the ground state of this sector. At $\lambda \to \infty$ it consists of a straight line of electric flux ($E_l = 1$) between the two charges. For finite but large values of λ its description will be complicated but its energy will have a term proportional to the extensive variable L. To be specific the energy of this state can be written as

$$E_{\text{string}} = AV + TL + B, \tag{5.7}$$

where V is the number of plaquettes in the lattice, L is the length defined in Eq. (5.6), and A, T, and B are calculable coefficients. A is the same constant appearing also in the energy of the vacuum (lowest state of the gauge-invariant sector)

$$E_{\text{vac}} = AV + C \tag{5.8}$$

representing the volume effect of all disconnected diagrams. The parameter T may then be defined as

$$T = \lim_{L\to\infty} \frac{E_{\text{string}} - E_{\text{vac}}}{L}, \tag{5.9}$$

and is known as the tension of the string. A value of λ at which T (or its derivative) shows a dis-

continuous behavior is a phase-transition point. The one of particular interest is where T vanishes. This signifies the transition from an electric-confining to nonconfining phase. At this point the vacuum includes arbitrarily long electric flux tubes. Below this point the tension stays at zero and loses its meaning.

The calculation of T is a tedious but straightforward exercise in perturbation theory. For high enough N (i.e., $\delta = 2\pi/N \ll 1$) we obtain

$$T \simeq \tfrac{1}{2}\lambda\delta^2 - \frac{0.167}{\lambda\delta^2} - \frac{0.102}{\lambda^3\delta^6}. \tag{5.10}$$

To continue this perturbation series outside its range of convergence we use the Padé approximation

$$T \simeq \frac{1}{2}\left(\lambda\delta^2 - \frac{0.94}{\lambda\delta^2}\right)\left(1 - \frac{0.61}{\lambda^2\delta^4}\right)^{-1}. \tag{5.11}$$

Note that to this order of perturbation theory T vanishes at $\lambda_c = 0.02N^2$.

In the weak-coupling regime we choose to study the mass of the monopole which is created by the operator $S_p^\dagger$. The monopoles are field configurations which exist in every sector of H. In order to isolate them we restrict ourselves to H_D of Eq. (4.9). $S_p^\dagger$ creates a topologically stable excitation in the theory defined by H_D. The reason is that the operator

$$\mathfrak{a}(W) = \prod_p R_p, \tag{5.12}$$

which is the Wilson loop around the whole lattice, commutes with H_D. This quantum number counts the total number of monopoles minus the total number of antimonopoles. Note that the boundary terms in Eq. (4.10) spoil the conservation of $\mathfrak{a}(W)$ in the theory defined by H. To the extent that the lattice is very large the boundary plays a negligible role. It is easy to see that to create a monopole at the center of the lattice in the weak-coupling vacuum one has to work to order $\frac{1}{2}\sqrt{V}$ in perturbation theory.

We define the mass of the monopole as

$$M_m(\lambda) = E_m(\lambda) - E_{\text{vac}}(\lambda), \tag{5.13}$$

where E_m is the energy of the state which develops with λ out of the state $\sum_p S_p^\dagger|0\rangle$ at $\lambda = 0$. Using perturbation theory it is easy to derive the first few terms of the mass

$$M_m \simeq (1 - \cos\delta) - \lambda,$$

which, for very high N turn into a series in λ/δ^2:

$$M_m = \tfrac{1}{2}\delta^2\left[1 - \frac{2\lambda}{\delta^2} + O\left(\frac{\lambda}{\delta^2}\right)^2\right]. \tag{5.14}$$

This will give positive results as long as λ is in a range of order N^{-2}. Throughout this region $\langle S_p\rangle_0$ must vanish. At the point where M_m vanishes S_p starts developing a vacuum expectation value. We expect this first phase-transition point on the λ scale to decrease as N^{-2} for large N. This is based on the perturbative expansion which can be regarded as an indication but not a proof.

In the limit $\lambda \to 0$, $N \to \infty$, our model turns into the XY model.[20] To see this let us use the representation

$$S_p = e^{i\phi_p}, \quad R_p = e^{-i\delta L_p}, \tag{5.15}$$

where ϕ and L play the roles of angle and angular momentum:

$$[L_p, \phi_q] = -i\delta_{p,q}. \tag{5.16}$$

The remarks concerning the E and A algebra in Sec. III and Appendix B apply here as well. With this identification H_D becomes

$$H_D = \lambda \sum_{\langle pq\rangle} [1 - \cos(\phi_p - \phi_q)] + \sum_p (1 - \cos\delta L_p), \tag{5.17}$$

which in the limit

$$\lambda \to 0, \quad \delta \to 0, \quad \delta^2/\lambda = \kappa^2 \tag{5.18}$$

reduces to

$$\frac{H_D}{\lambda} \simeq \sum_{\langle pq\rangle} [1 - \cos(\phi_p - \phi_q)] + \tfrac{1}{2}\kappa^2 \sum_p L_p^2. \tag{5.19}$$

This XY model has two phases[21]: At very low λ (high κ^2) it is in the disordered phase which has a unique vacuum and a mass gap (massive monopole), whereas at high λ (low κ^2), ϕ develops a vacuum expectation value and the system moves into an ordered Goldstone phase.

A question of prime importance is whether the phase-transition point where M_m vanishes as one increases λ from below coincides with the point where T vanishes as one decreases λ from above. A positive answer will mean that the theory has two phases, whereas a negative answer implies that this model has at least three phases. Perturbation theory indicates that for very high N the lower critical point behaves like N^{-2}, whereas the higher critical point increases like N^2. We will encounter a similar behavior in $D = 3$ dimensions and will argue there that it means that from some critical N value the model turns from a two-phase to a three-phase system. However, in $D = 2$ space dimensions it is known that such a perturbative argument fails in a model which is related to our problem. We have already mentioned that as $N \to \infty$ and $\lambda \sim N^2$ our model turns into PQED.[5] This relation will be investigated in Sec.

VII. Whereas perturbative arguments would lead to the vanishing of the tension of PQED for a finite coupling constant (i.e., finite value of λN^{-2}), it is known that the $D=2$ case has only one phase due to nonperturbative effects.[5,22]

VI. SELF-DUALITY OF THE $Z(N)$ MODEL IN THREE SPACE DIMENSIONS

We turn now to the self-dual properties of our Hamiltonian in three space dimensions. The dual transformations replace link variables by plaquette variables and vice versa. Since the link variable in Eq. (3.16) is E_l and the plaquette variable is B_p their physical reinterpretation will be the interchange of the electric and magnetic fields. We started by defining our theory in terms of the link variables P_l and Q_l whose algebraic properties are defined by Eq. (3.2) and (3.3). As in Sec. IV we now define a new set of operators R_p and S_p associated with the plaquettes of the same lattice and obeying the same set of algebraic conditions as P_l and Q_l. We require the following connection between the two sets of operators:

$$R_p^\dagger = Q_{p1}^\dagger Q_{p2}^\dagger Q_{p3} Q_{p4}\,, \tag{6.1}$$

$$P^\dagger = S_{l1}^\dagger S_{l2}^\dagger S_{l3} S_{l4}\,. \tag{6.2}$$

Equation (6.1) coincides with Eq. (4.3) in the $D=2$ case. Equation (6.2) differs from the corresponding Eq. (4.1) and involves four plaquette operators. The notation used in Eq. (6.2) is explained in Fig. 6, which is the dual to Fig. 1. It is clear by inspection that the $Z(N)$ algebra rules of the $P_l Q_l$ set and the $R_p S_p$ set are consistent with one another. In Appendix C we describe an explicit solution of the S_p operators in terms of the P_l operators.

The gauge condition $G(i)=1$ is automatically satisfied by Eq. (6.2). This is therefore a solution appropriate for the gauge-invariant sector only. In contradistinction with the $D=2$ case we have not obtained a set of independent operators. This is indicated by the fact that the number of new variables is the same as before (apart from the surface terms discussed in Appendix C). Equation (6.1) implies that for every cube the product over its enclosing plaquette operators is constrained as follows:

$$W(c) = \prod_{p_+ \ni c} R_p^\dagger \prod_{p_- \ni c} R_p = 1\,. \tag{6.3}$$

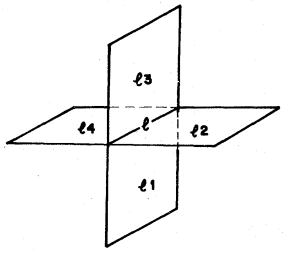

FIG. 6. Notation of plaquettes around a link.

The notation in Eq. (6.3) is the dual lattice variation of Fig. 2. What happened can be easily understood in terms of the electromagnetic variables. We note that Eq. (3.1) now takes the dual form

$$H_D = -\tfrac{1}{2}\lambda \sum_l (S_{l1}^\dagger S_{l2}^\dagger S_{l3} S_{l4} + \text{H.c.} - 2) - \tfrac{1}{2}\sum_p (R_p^\dagger + R_p - 2)\,, \tag{6.4}$$

which suggests the identification

$$R_p = e^{i\delta \tilde{E}_p}\,, \quad S_p = e^{i\tilde{A}_p}\,, \quad \tilde{B}_l = (\nabla \times \tilde{A})_l\,, \tag{6.5}$$

where we now choose the quantum-mechanical operators to be $\tilde{E}_p$ and $e^{i\tilde{A}_p}$. This leads to the following form of our Hamiltonian:

$$H_D = \lambda \sum_l (1 - \cos \tilde{B}_l) + \sum_p (1 - \cos \delta \tilde{E}_p)\,. \tag{6.6}$$

Comparing Eq. (6.6) with Eq. (3.16) we find the changes

$$E_l \to \delta^{-1} \tilde{B}_l\,, \quad B_p \to \delta \tilde{E}_p\,. \tag{6.7}$$

Since by definition $\tilde{B}_l$ is divergenceless we are in the gauge-invariant sector. However, in order for $\tilde{E}_p$ to represent the same physics as B_p, it should obey the $\nabla \cdot \tilde{E} = 0$ condition which is the same as Eq. (6.3).

In the $D=3$ case the dual transformations lead to a new interesting feature which is known as self-duality. The Hamiltonian H_D of Eq. (6.4) has similar structure to H of Eq. (3.1). The plaquette operators in H_D obey the same algebra as the link operators in H and, therefore, the physical contents of the theory defined by $H_D(\lambda)$ are the same as those of $\lambda H(\lambda^{-1})$. In addition, we know that $H_D(\lambda)$ is equivalent to $H(\lambda)$ in the gauge-invariant sector. Therefore, every physical quantity which is linearly dependent on the Hamiltonian must obey the self-duality relation

$$F(\lambda) = \lambda F(\lambda^{-1})\,. \tag{6.8}$$

In particular the phase diagram of this model should exhibit a symmetry under the inversion $\lambda \to \lambda^{-1}$.

We should remark that in the dual formulation of the Hamiltonian we did not take proper care of the boundary terms which appear once the inversion of Eq. (6.2) is carried out. These terms are exhibited and discussed in Appendix C. We assume that they do not affect the phase structure of the theory and have therefore kept only the

volume terms in Eq. (6.4) and (6.6).

As for the $D=2$ case, we can identify the order and disorder operators and check their algebra. The order parameter is, as before, $\langle \mathfrak{A}(C)\rangle_0$. It obeys the area and perimeter laws at very large and very small λ values, respectively, as described in Eqs. (5.4) and (5.5). The definition of the disorder loop operator $\mathfrak{B}(C)$ is dual to that of $\mathfrak{A}(C)$ in Eq. (4.18):

$$\mathfrak{B}(C)=\prod_{p_D\in\Sigma} P_{p_D}. \tag{6.9}$$

The curve C is defined on the dual lattice. The product is over all p_D plaquettes on the dual lattice whose union forms a surface Σ enclosed by the curve C. In terms of the original lattice the p_D are links intersecting the surface Σ.

Self-duality of the theory implies that $\langle \mathfrak{B}(C)\rangle_0$ has the complementary behavior to $\langle \mathfrak{A}(C)\rangle_0$:

$$\langle \mathfrak{B}(C)\rangle_0 \sim \begin{cases} \exp[-\text{perimeter }(C)], & \lambda \gg 1 \\ \exp[-\text{area }(C)], & \lambda \ll 1. \end{cases} \tag{6.10}$$

Using the $Z(N)$ algebra of our operators we find that $\mathfrak{A}(C)$ and $\mathfrak{B}(C')$ obey the 't Hooft topological algebra

$$\mathfrak{A}^\dagger(C)\mathfrak{B}(C')\mathfrak{A}(C)=e^{i m\delta}\mathfrak{B}(C'), \tag{6.11}$$

where $m=0$ if C and C' do not wind around one another and $m=1$ if they do. This can be generalized to complicated intertwining curves with m representing the winding number. For this purpose it is easier to replace Eq. (6.9) with a definition in terms of products of S and $S^\dagger$ operators along the curve C' on the dual lattice. Using Eq. (6.2) we find

$$\mathfrak{B}(C')=\prod_{l_+\in C'} S_l \prod_{l_-\in C'} S_l^\dagger, \tag{6.12}$$

where l_+ and l_- are positive and negative links on the dual lattice along the curve C'. This is the analog of a line integral representation. It is now easy to see that if C in Eq. (6.11) lies in a plane one can recover the $D=2$ result, Eq. (4.19), in this plane.

VII. PHASES OF THE $Z(N)$ THEORY IN THREE SPACE DIMENSIONS

The self-duality of our Hamiltonian in $D=3$ dimensions suffices to locate the phase-transition point if it is assumed that the theory has two phases only. The inversion symmetry under $\lambda\to\lambda^{-1}$ implies that this point has to be $\lambda=1$. This is believed to be the situation for $N=2$.[15] We will argue that this cannot be the case for all N. In particular we will indicate that for high N more than two phases exist. Once the two-phase assumption is relaxed, self-duality by itself can no longer determine the locations of the phase-transition points.

The physical quantity which we investigate in the strong-coupling (high-λ) regime is the string tension of the electric flux tube as defined in Eq. (5.9). A phase transition occurs at that λ_c where the tension vanishes. This point is also a zero of the renormalization-group β function. Using this concept together with self-duality we can show that at least one such zero should occur. For a lattice spacing a let us define the Hamiltonian H_a:

$$H_a=-\tfrac{1}{2}\alpha\lambda\sum_l (P_l^\dagger+P_l-2)-\tfrac{1}{2}\alpha\sum_p (R_p^\dagger+R_p-2), \tag{7.1}$$

where α and λ are the common scale factor and the relative strength for the given scale a. The a dependence of λ determines the β function:

$$\beta=a\partial\lambda/\partial a. \tag{7.2}$$

This function appears in renormalization-group equations which state that physical quantities stay constant under changes of the scale of the underlying lattice. Such a quantity is the energy of a string. Suppose the string has a dimensional length L. We define the string tension as

$$E_{\text{string}}-E_{\text{vacuum}}=(L/a)T. \tag{7.3}$$

L stays constant and T changes as a is being changed. Requiring this energy difference to stay constant we find

$$T=\beta\frac{\partial T}{\partial\lambda}+\frac{\partial T}{\partial\alpha}\frac{\partial\alpha}{\partial a}. \tag{7.4}$$

The naive dimension of α is that of energy, i.e., a^{-1}. Suppose its anomalous dimension is γ, i.e., $\alpha\sim a^{-1-\gamma}$. Using this as well as the fact that α appears as a simple multiplicative factor in T, we can rewrite Eq. (7.4) as

$$\beta=(2+\gamma)\frac{T}{\partial T/\partial\lambda}. \tag{7.5}$$

This relation is valid for high λ where the electric flux tension is well defined. In the limit $\lambda\to\infty$ perturbation theory tells us that $T\sim\lambda$, thus implying that $\beta\sim\lambda$. We will show that self-duality leads then to the conclusion that near $\lambda=0$ one obtains $\beta\sim-\lambda$.

The parameters α and λ depend on a and on initial values α_0 and λ_0 chosen at some scale $a=a_0$:

$$\alpha=\alpha(\alpha_0,\lambda_0,a),\quad \lambda=\lambda(\alpha_0,\lambda_0,a). \tag{7.6}$$

Self-duality leads to the relations

$$\alpha(\alpha_0, \lambda_0, a) = \lambda_1^{-1}\alpha(\alpha_0\lambda_0, \lambda_0^{-1}, a), \tag{7.7}$$

$$\lambda_1 = \lambda(\alpha_0, \lambda_0, a) = [\lambda(\alpha_0\lambda_0, \lambda_0^{-1}, a)]^{-1}. \tag{7.8}$$

Taking the derivative of Eq. (7.8) with respect to a we are led to

$$\lim_{\lambda\to 0} \beta(\lambda) = \lim_{\lambda\to 0} (-\lambda^2)\beta(\lambda^{-1}). \tag{7.9}$$

Therefore if $\lim_{\lambda\to\infty}\beta(\lambda) = c\lambda$ then $\lim_{\lambda\to 0}\beta(\lambda) = -c\lambda$. Clearly any continuous β function with such behavior at the end points of the λ scale has to vanish at least once in between. Hence this theory possesses at least two phases. If there are only two phases β vanishes at $\lambda_c = 1$. That is also where the electric tension vanishes. For $\lambda < 1$ one encounters magnetic tension which is the dual phenomenon to electric tension. We will present arguments that for large values of N the electric tension vanishes at a high value of $\lambda_c \sim N^2$, and return later to a discussion of the possible form of the β function in our theory.

In Sec. V we have defined the electric string tension. The same definition holds also in the three-dimensional problem. Using simple perturbation theory we calculate the tension as a series in λ^{-1} for very high N, i.e., $\delta = 2\pi/N \ll 1$. The result is

$$T \simeq \tfrac{1}{2}\lambda\delta^2 - \frac{0.333}{\lambda\delta^2} - \frac{0.290}{\lambda^3\delta^6}. \tag{7.10}$$

Rewriting it in a Padé form one obtains

$$T \simeq \frac{1}{2}\left(\lambda\delta^2 - \frac{1.537}{\lambda\delta^2}\right)\left(1 - \frac{0.870}{\lambda^2\delta^4}\right)^{-1}, \tag{7.11}$$

which has a zero at $\lambda_c = 0.031N^2$ for $N \gg 2\pi$.

The argument for $\lambda_c \sim N^2$ can be made stronger by noticing the connection between the $Z(N)$ theories and PQED: In the limit

$$N\to\infty, \quad \lambda\to\infty, \quad \lambda\delta^2 = e^4 = \text{const}, \tag{7.12}$$

the Hamiltonian (3.16) becomes

$$H = \tfrac{1}{2}e^4 \sum_l E_l^2 + \sum_p (1 - \cos B_p) + e^4 O\left(\delta^2 \sum_l E_l^2\right). \tag{7.13}$$

Neglecting the $\delta^2 E_l^2$ and higher terms since $\delta \to 0$, we are left with the PQED Hamiltonian

$$\frac{H}{e^2} = \tfrac{1}{2}\left[\sum_l \bar{E}_l^2 + \frac{2}{e^2}\sum_p (1 - \cos e\bar{B}_p)\right]. \tag{7.14}$$

$\bar{E}_l$ and $\bar{B}_p$ are canonical transformations of the original electric and magnetic fields:

$$\bar{E}_l = eE_l, \quad \bar{B}_p = e^{-1}B_p. \tag{7.15}$$

The PQED theory of Eq. (7.14) is known to have a phase transition for a finite value of e. For $e > e_c$ it exhibits a confinement of electric charges and a formation of electric flux lines. The perturbative equivalence between the high-N limit of the $Z(N)$ models and PQED leads to the suggestion that the high-λ_c value of $Z(N)$ approaches $e_c^4\delta^{-2}$ as $N\to\infty$.

For $e < e_c$ PQED has a nonconfining phase. As $e \to 0$ it looks more and more like normal QED. Such a nonconfining phase follows also in the $Z(N)$ theories throughout the regions of constant λ as $N\to\infty$. This can be simply seen by letting $e \to 0$ in Eq. (7.14). However, if one goes into the other extreme limit

$$N\to\infty, \quad \lambda\to 0, \quad \delta^2/\lambda = g^4 = \text{const}, \tag{7.16}$$

one finds that the theory is perturbatively equivalent to

$$\frac{H}{\lambda g^2} \simeq \tfrac{1}{2}\left[\sum_p \tilde{B}_p^2 + \frac{2}{g^2}\sum_l (1 - \cos g\tilde{E}_l)\right], \tag{7.17}$$

where the new electric and magnetic fields are

$$\tilde{E}_l = g\sqrt{\lambda}\, E_l, \quad \tilde{B}_p = \frac{B_p}{g\sqrt{\lambda}}. \tag{7.18}$$

Equation (7.17) is once again a PQED theory but this time the electric and magnetic fields are interchanged. This was to be expected in view of the self-duality of the $Z(N)$ theories. At a new critical point whose value is $\lambda_{c'} = \lambda_c^{-1}$, we find a transition between the nonconfining phase to a magnetic-confining one. In this new phase magnetic flux is confined to flux tubes in much the same way that the electric flux was confined in the high-λ region. However, in view of the overall factor of λ, the magnetic flux loops will in fact have very low energies.

The considerations of the previous paragraphs lead us to expect that the $Z(N)$ theories have a phase diagram of the type shown in Fig. 7. For low N there are two phases. Above some N_c a three-phase structure develops. To estimate the value of N_c we return to the perturbative calculation of the electric tension. A calculation to second order in λ^{-1} without any approximations

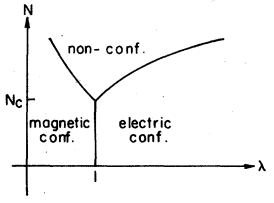

FIG. 7. The phase diagram of $Z(N)$ models.

for small δ leads to

$$T = \lambda(1-\cos\delta) - \frac{1}{2\lambda(1-\cos\delta)+\lambda(1-\cos2\delta)} . \quad (7.19)$$

This expression vanishes at $\lambda_c = \lambda_c^{(2)}$. Our experience with $Z(2)$ as well as with the large-N approximation in Eq. (7.10) shows us that the fourth-order estimate of the zero point is higher, $\lambda_c^{(4)} > \lambda_c^{(2)}$. Assuming this to be an indication that $\lambda_c > \lambda_c^{(2)}$ we ask which N is the first to have $\lambda_c^{(2)} > 1$. The answer is $N = 7$. We conclude, therefore, on the basis of second-order perturbation theory, that $N_c \lesssim 7$.

The vacuum of the magnetic-confining (MC) phase is a condensate of electric flux tubes.[23] Similarly the vacuum of the electric-confining (EC) phase is a condensate of magnetic flux tubes.[24] In the nonconfining (NC) phase both kinds of flux tubes exist in the vacuum and neither is confined. In this region both the electric and magnetic descriptions hold. Comparing Eq. (7.12) and (7.16) we find the relation between the two coupling constants

$$eg = 2\pi/N . \quad (7.20)$$

This is different from the Dirac quantization condition. We can, however, identify magnetic monopole configurations which do obey Dirac's condition. This is a configuration in which N magnetic flux units emerge from a finite region on the lattice, and therefore its total magnetic strength is $Ng = 2\pi/e = g_D$. This is possible since the flux is conserved only modulo N. These flux lines have to meet again either on the boundary or in some other region of the lattice which can be identified as an antimonopole (see an example in Fig. 8). In the MC phase we can observe at most magnetic monopole-antimonopole pairs. In the NC phase, where the magnetic tension vanishes, the pairs can have arbitrarily long strings, i.e., the monopoles are liberated. Similarly there exist electric monopoles with charge Ne which are possible excitations in the gauge-invariant sector. They exist only as bound states in the EC phase and are liberated in the NC phase.

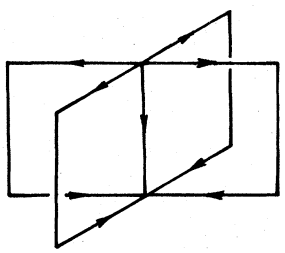

FIG. 8. A monopole-antimonopole configuration. If each line carries one flux unit the depicted configuration is allowed for $N = 5$.

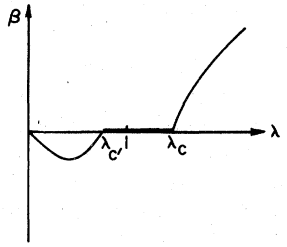

FIG. 9. Expected behavior of the β function defined in Eq. (7.2) for $N > N_c$.

Using an analogy with superconductivity we may speculate that at the transition point from the NC to EC phase, the magnetic monopoles become massless and the EC vacuum becomes their condensate.[25] This scenario is known to hold true in PQED,[21] and is very suggestive that it holds for finite N, too.[26] Duality implies then that the MC vacuum becomes a condensate of the electric monopoles of charge Ne, thus confining magnetic flux units of charge g.

Finally we would like to discuss the form of the β function in our model. We know its behavior at the two end points of the λ scale. If it is to describe a system with three phases then the only possible choice which reflects the self-dual character of the theory is the form depicted in Fig. 9. In the nonconfining region $\beta = 0$ as in pure QED because of the existence of massless photons in this phase.

It is interesting to note that such a behavior was found by Jose *et al.*[27] in their investigation of global $Z(N)$ spin models in $1+1$ dimensions using the Migdal approximation. Other type of approximations also revealed this behavior.[28] This is relevant to our problem since Migdal has argued[29] that gauge theories in $3+1$ dimensions have the same behavior as their $1+1$ counterpart spin models.

Upon completion of our work we received a report by Cardy[30] proving that the global $Z(N)$ spin models in $1+1$ dimensions have three phases for $N \gtrsim 5$. He has also obtained this form of the β function. We have been informed[26] that a similar proof using the action approach was obtained by Elitzur *et al.*[31] for both the $1+1$ global $Z(N)$ and the $3+1$ local $Z(N)$ gauge theories.

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APPENDIX A: DUAL TRANSFORMATIONS IN FOUR-DIMENSIONAL $Z(N)$ MODELS[32]

In this Appendix we will discuss the algebraic aspects of the self-duality conditions within the action approach.[33] Using the statistical-mechanics terminology we define a partition function on a four-dimensional lattice:

$$Z=\sum_{\{U_{ij}\}}\exp\left[\sum_{p} f(U_p)\right], \tag{A1}$$

where the $Z(N)$ variables are defined as

$$U_{ij}=U^{-1}{}_{ji}=e^{(2\pi i/N)n_{ij}}, \quad n_{ij}=\text{integer} \tag{A2}$$

and the plaquette variable U_p is

$$U_p=U_{ij}U_{jk}U_{kl}U_{li}, \tag{A3}$$

with i, j, k, l being the four vertices defining the plaquette p. The function $f(U_p)$ is a finite power series:

$$f(U)=\sum_{m\in Z^+(N)} c_m U^m, \quad e^{f(U)}=\sum_{m\in Z^+(N)} d_m U^m, \tag{A4}$$

where the summation is over integers mod N. This follows of course from the fact that $U\in Z(N)$.

In four-dimensional $Z(N)$ models one finds a similar description of the same theory in terms of dual variables V_{ij} on the dual lattice:

$$Z=\sum_{\{V_{ij}\}}\exp\left[\sum_{p} g(V_p)\right], \tag{A5}$$

$$g(V)=\sum_{m\in Z^+(N)} \tilde{c}_m V^m, \quad e^{g(V)}=\sum_{m\in Z^+(N)} \tilde{d}_m V^m. \tag{A6}$$

The connection between the two series of Eq. (A4) and (A6), i.e., the duality relation, is[7,16]

$$d_m=\sum_n \tilde{d}_n e^{(2\pi i/N)mn}, \quad m, n, \in Z^+(N). \tag{A7}$$

The interesting question is whether the Wilson action

$$f_W=U+U^+ \tag{A8}$$

is self-dual or not. The function $f(U)$ is both Hermitian and real, i.e.,

$$f(U)=f^*(U), \tag{A9}$$

$$f(U)=f^*(U^*). \tag{A10}$$

Using these two properties in the duality relation, Eq. (A7), we learn that they imply

$$g(U)=g^*(U^*), \tag{A11}$$

$$g(U)=g^*(U), \tag{A12}$$

respectively. The set of the two conditions stays therefore invariant under the duality transformation. In terms of the power-series coefficients they mean

$$c_m=c_m^*=c_{-m} \Rightarrow \tilde{c}_m=\tilde{c}_{-m}=\tilde{c}_m^*. \tag{A13}$$

Without any loss of generality we may choose $c_0=0$. This leaves us with $N-1$ real parameters on which we can impose the condition (A13). In a $Z(2)$ model there exists only one free parameter and the most general action is the Wilson one. However, also in $Z(3)$ these conditions leave only one real parameter $c_1=c_2$. In other words the two elements $e^{\pm 2i\pi/3}$ have the same free energy. Once again the only remaining term is the action (A8).

For higher N values the constraints of Eq. (A13) are no longer sufficient to single out the Wilson action of Eq. (A8). Nevertheless, for every N it is possible to find a particular action structure which stays self-dual, i.e.,

$$\tilde{c}_m:\tilde{c}_n=c_m:c_n, \quad m, n\neq 0. \tag{A14}$$

This defines a specific line in the space of parameters (coupling constants) along which self-duality holds. For $N=4$ it turns out that this line still coincides with the Wilson action; however, for $N\geq 5$ it involves more complicated terms, i.e., $m\neq\pm 1$.

APPENDIX B: THE E AND A ALGEBRA

The unitary operators P and Q which obey the algebra defined by Eqs. (3.2) and (3.3) can be rewritten in terms of Hermitian operators E and A. Following Schwinger[17] we start with the representation

$$P=e^{i\eta E}, \quad Q=e^{i\eta A}, \quad \eta=\sqrt{\delta}=\left(\frac{2\pi}{N}\right)^{1/2}. \tag{B1}$$

The spectrum of both E and A can be chosen as

$$E_k=\eta k, \quad A_k=\eta k, \quad k\in\left[-\tfrac{1}{2}(N-1), -\tfrac{1}{2}(N-2), \ldots, \tfrac{1}{2}(N-1)\right], \tag{B2}$$

and, therefore, as $N\to\infty$

$$\max(E)=\max(A)\to\infty, \quad \Delta E=\Delta A=\eta\to 0. \tag{B3}$$

The representation (B1) is the convenient one for

obtaining noncompact operators E and A in the limit $N\to\infty$. This is the suitable choice for λ = const or $e\to 0$ in Eq. (7.14).

In an explicit realization in which P is diagonalized Q serves as a step operator:

$$P|u^k\rangle = e^{2\pi ik/N}|u^k\rangle\,,\quad Q|u^k\rangle = |u^{k-1}\rangle\,, \tag{B4}$$

where k belongs to the set defined in Eq. (B2). Alternatively one may think of k as the set of all integers (or half integers) and choose $u = e^{i\delta}$ so that there are only N states in the Hilbert space. Similarly when Q is diagonalized P becomes a step operator:

$$Q|v^k\rangle = e^{2\pi ik/N}|v^k\rangle\,,\quad P|v^k\rangle = |v^{k+1}\rangle\,. \tag{B5}$$

The cyclic property of $Z(N)$ is reflected through the operation of P and Q on the states which correspond to the end points of the spectrum:

$$\begin{aligned} &P|v^{(N-1)/2}\rangle = |v^{-(N-1)/2}\rangle\,,\\ &Q|u^{-(N-1)/2}\rangle = |u^{(N-1)/2}\rangle\,. \end{aligned} \tag{B6}$$

We use here Schwinger's notation. He shows that the two sets of states are related by

$$\langle u^k|v^l\rangle = \frac{1}{\sqrt{N}}e^{2\pi ikl/N}\,, \tag{B7}$$

and proceeds to define quantum-mechanical wave functions which we can represent as

$$\psi(E_k) = \frac{1}{\sqrt{\eta}}\langle u^k|\Psi\rangle\,,\quad \psi(A_k) = \frac{1}{\sqrt{\eta}}\langle v^k|\Psi\rangle\,. \tag{B8}$$

This definition guarantees that

$$\langle\Psi|\Psi\rangle = \sum_k \Delta E|\psi(E_k)|^2 = \sum_k \Delta A|\psi(A_k)|^2\,, \tag{B9}$$

where $\Delta E = \Delta A = \eta$. Going to the $N\to\infty$ limit in a proper way with suitable wave functions, one is led to a representation which supports the operator commutation relation

$$[E,A] = i\,. \tag{B10}$$

For every finite N this is not strictly true. Even the weaker relation

$$[E, e^{\pm iA}] = e^{\pm iA} \tag{B11}$$

is spoiled by the periodicity property Eq. (B6). However, as $N\to\infty$ we imagine the wave function to be concentrated far from the end points so that the periodic character drops out from the consideration.

In most of our applications of this algebra we use a representation which is different from Eq. (B1). The limit $\lambda\to\infty$, $N\to\infty$, $\lambda\delta^2$ = const is best described by using the representation

$$P = e^{i\delta E}\,,\quad Q = e^{iA}\,. \tag{B12}$$

For every finite N this amounts to a canonical transformation of the operators E and A. Their range can now be chosen as

$$\begin{aligned} &E\in[-\tfrac{1}{2}(N-1)\,,\ -\tfrac{1}{2}(N-2),\ldots,\tfrac{1}{2}(N-1)]\,,\\ &A\in\delta[-\tfrac{1}{2}(N-1)\,,\ -\tfrac{1}{2}(N-2),\ldots,\tfrac{1}{2}(N-1)]\,, \end{aligned} \tag{B13}$$

and the $N\to\infty$ limit leads to

$$\begin{aligned} &E\in[-\infty\,,\infty]\,,\quad \Delta E = 1\\ &A\in[-\pi\,,\pi]\,,\quad \Delta A = \delta\to 0\,. \end{aligned} \tag{B14}$$

In this limit the operator A is bounded and has a continuous spectrum, whereas the operator E is unbounded and has a discrete spectrum. The commutation relation (B10) will fail near the end points of the support of A. In fact, since A acquires the operator properties of an angle, it becomes ill defined because its basis is the set of periodic wave functions. One should indeed consider only e^{iA}. Its commutation relation with E, Eq. (B11), will be valid in the $N\to\infty$ limit for a space of functions which are smooth in E and vanish at infinity.

As an example let us calculate $[E,A]$ using the identification of Eq. (B12). E and A can now be written as

$$E = \sum_k k|u^k\rangle\langle u^k|\,,\quad A = \frac{2\pi}{N}\sum_k k|v^k\rangle\langle v^k|\,. \tag{B15}$$

Using Eq. (B7) we find that

$$\lim_{(l-m)/N\to 0}\langle u^l|[E,A]|u^m\rangle = i\delta_{l,m} - i(-1)^{l-m}\,. \tag{B16}$$

In defining wave functions which are normalizable in the sense of Eq. (B9) we have to replace (B8) by the new definitions

$$\phi(E_k) = \langle u^k|\Psi\rangle\,,\quad \phi(A_k) = \delta^{-1/2}\langle v^k|\Psi\rangle\,. \tag{B17}$$

Here $\phi(E_k)$ is defined for discrete values and $\Delta E_k = 1$. For every finite N it is, however, equal to $\delta^{1/4}\psi(E_k)$ of Eq. (B8). It follows that in order to be able to connect the compact QED formulation with the noncompact one, $\phi(E_k)$ must be a very slowly varying function of its argument. For such a function we will find

$$\begin{aligned} \langle\Psi'|[E,A]|\Psi\rangle &= \sum_{l,m}\phi'^*(E_l)\phi(E_m)[i\delta_{l,m} - i(-1)^{l-m}]\\ &\approx i\langle\Psi'|\Psi\rangle\,. \end{aligned} \tag{B18}$$

We see that the $Z(N)$ formalism is useful in giving a meaning to the transformation between the compact and noncompact versions of QED.

APPENDIX C: CONSTRUCTION OF DUAL VARIABLES IN THREE DIMENSIONS

We associate with each point of the lattice (i,j,k) three links which are parallel to the basis vectors. The P_l variables which correspond to these links are denoted by $P_x(i,j,k)$, $P_y(i,j,k)$, and $P_z(i,j,k)$. Examples of this are shown in Fig. 10 which displays the cubic lattice in question. The gauge condition which specifies the gauge-invariant sector then takes the form

$$P_x(i,j,k)P_x^\dagger(i-1,j,k)P_y(i,j,k)P_y^\dagger(i,j-1,k) \times P_z(i,j,k)P_z^\dagger(i,j,k-1)=1\,. \quad \text{(C1)}$$

This is true everywhere inside the lattice but not near the surface. Thus if we choose the planes $x=0$, $y=0$, and $z=0$ to be three out of the six surface planes we find gauge conditions such as

$$P_x(0,0,0)P_y(0,0,0)P_z(0,0,0)=1\,, \quad \text{(C2)}$$
$$P_x(1,0,0)P_x^\dagger(0,0,0)P_y(1,0,0)P_z(1,0,0)=1\,, \text{ etc.},$$

i.e., whenever one of the links of Eq. (C1) does not exist the corresponding variable drops out from the gauge condition. We may choose one P_l link variable at each vertex to be dependent on all the other link variables. This is analogous to the choice of a gauge. We will henceforth choose $P_z(i,j,k)$ to be that dependent variable. The independent variables are therefore defined on the $z=$ const planes of the lattice.

Let us turn to an explicit construction of the dependent $P_z(i,j,k)$ variables. We start from one end of the lattice, e.g., $k=0$. It follows then from the gauge condition that

$$P_z(i,j,0)=P_x(i-1,j,0)P_x^\dagger(i,j,0) \times P_y(i,j-1,0)P_y^\dagger(i,j,0)\,, \quad \text{(C3)}$$

and, as mentioned above, whenever a link is not defined (near the edges) one inserts the number 1 for the corresponding variable. The next plane of P_z variables will be given by

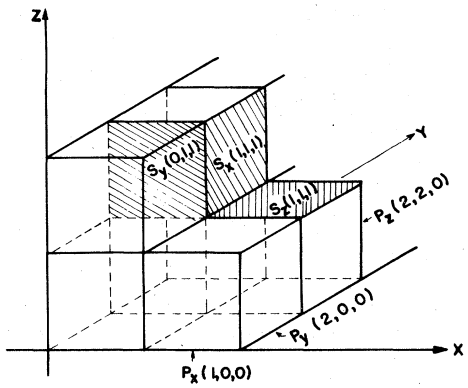

FIG. 10. Notation of links and plaquettes on a three-dimensional lattice.

$$P_z(i,j,1)=P_x(i-1,j,1)P_x^\dagger(i,j,1)P_y(i,j-1,1) \times P_y^\dagger(i,j,1)P_z(i,j,0)\,, \quad \text{(C4)}$$

and for $P_z(i,j,0)$ one can insert Eq. (C3). This construction leads to the result

$$P_z(i,j,k)=\prod_{k'=0}^{k} P_x(i-1,j,k')P_x^\dagger(i,j,k') \times P_y(i,j-1,k')P_y^\dagger(i,j,k')\,, \quad \text{(C5)}$$

which is an explicit construction of all P_z variables. It should be noted that in a finite lattice $(0\le k\le K)$ the P_x and P_y variables in the last plane $(k=K)$ are not all free, and one has then to make an additional choice, e.g., taking all $P_y(i,j,K)$ to be dependent variables.

We can now make contact with the plaquette variables S_p defined in Eq. (6.2). Let us use a plaquette notation similar to that of the link notation. One associates three plaquettes with each point (i,j,k) and defines the operators in the following way:

$$\begin{aligned} &S_x(i,j,k) \text{ on the plaquette } (i,j,k)(i,j+1,k)(i,j+1,k+1)(i,j,k+1)\,,\\ &S_y(i,j,k) \text{ on the plaquette } (i,j,k)(i,j,k+1)(i+1,j,k+1)(i+1,j,k)\,,\\ &S_z(i,j,k) \text{ on the plaquette } (i,j,k)(i+1,j,k)(i+1,j+1,k)(i,j+1,k)\,. \end{aligned} \quad \text{(C6)}$$

Examples of this are given in Fig. 10. Using now the explicit realization

$$S_x(i,j,k)=\prod_{k'=0}^{k} P_y^\dagger(i,j,k')\,,$$
$$S_y(i,j,k)=\prod_{k'=0}^{k} P_x^\dagger(i,j,k')\,, \quad \text{(C7)}$$

we see that Eq. (C5) can be rewritten as

$$P_z(i,j,k)=S_x(i,j,k)S_x^\dagger(i,j-1,k) \times S_y(i,j,k)S_y^\dagger(i-1,j,k)\,. \quad \text{(C8)}$$

This form is completely consistent with Eq. (6.2). In order to satisfy the other two equations contained in Eq. (6.2), namely

$$P_y(i,j,k) = S_x^\dagger(i,j,k) S_x(i,j,k-1) \times S_z(i,j,k) S_z^\dagger(i-1,j,k)\,, \tag{C9}$$

$$P_x(i,j,k) = S_y^\dagger(i,j,k) S_y(i,j,k-1) \times S_z(i,j-1,k) S_z^\dagger(i,j,k)\,,$$

we have to add to Eq. (C7):

$$S_z(i,j,k) = \text{const}\,. \tag{C10}$$

Thus we see that choosing P_z to be the dependent variable was equivalent to the axial-gauge choice $S_z = \text{const}$.

The number of independent S_x and S_y variables is the same as the number of independent P_y and P_x variables. For a finite lattice there will appear additional constraints. Thus if $k_{\max} = K$ one finds for each (i,j,k)

$$S_x(i,j,K) S_x^\dagger(i,j-1,K) S_y(i,j,K) S_y^\dagger(i-1,j,K) = 1\,, \tag{C11}$$

which was already mentioned above after Eq. (C5).

Equations (C9) get modified on the surfaces of the lattice. At $k=0$ one term (namely S_x or S_y at $k-1$) does not exist and should be replaced by the number 1. Hence when one writes H in terms of the dual (plaquette) variables one encounters surface terms which are different from the usual volume terms—they may depend only on three or even two (at the edges) S_p variables instead of the usual four—S_p form. When going to the electromagnetic-type variables one can replace these surface terms by boundary conditions. Thus one can define B_l on the $k=0$ lattice plane but require that $A_x = A_y = 0$ on the $k=-1$ level.

The effect of the surface terms was displayed explicitly in the two-dimensional case in Eq. (4.10). Their structure and role in the $D=3$ case is similar. They remove the degeneracy of the vacuum in the high-λ region but should not affect the character of the phase transition on a large lattice.

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