GEOMETRIC FERMIONS ☆

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We study the Kähler–Dirac equation which linearizes the laplacian on the space of antisymmetric tensor fields. In flat space–time it is equivalent to the Dirac equation with internal symmetry and on the lattice it reproduces Susskind fermions. The KD equation in curved space–time differs from the Dirac equation by coupling the gravitational field to the internal symmetry generators. This new way of treating fermionic degrees of freedom may lead to a solution of the generation puzzle but is in conflict with the equivalence principle and with Lorentz invariance on the Planck-mass scale.

In the early 1960's, the mathematician E. Kähler [1] introduced a transcription of the Dirac equation as a set of equations for antisymmetric tensor fields. This Kähler–Dirac (KD) equation has been largely ignored by physicists, but has recently been studied by several authors in connection with lattice fermions [2,3]. Indeed, as we will show below, KD fields are a natural framework for understanding Susskind's [4] lattice fermions. Moreover, the KD equation may be generalized to any riemannian or pseudo-riemannian manifold. This was shown by Graf [5] who suggested that KD fields might be more fundamental than Dirac spinors. This is an appealing idea because it conforms to Einstein's philosophy of associating all physical fields with geometrical objects. It has one immediate consequence that we are familiar with from lattice models: the appearance of replicas of spinor fields with identical quantum numbers.

We take up the idea that Kähler's geometric fermions are the fundamental fermi fields, and investigate the possibility that the replication is associated with the generation structure of the fermion spectrum. We show that the KD equation in a gravitational field deviates from the conventional generalization of the Dirac equation and argue that it may play an important role in fixing the fermion mass matrix. The KD equation leads to the breakdown of Lorentz invariance on the scale of the Planck-mass where gravitational quantum fluctuations become important.

The KD operator on an n-dimensional riemannian [‡1] manifold with metric $g_{\mu\nu}$ operates in the space of all covariant antisymmetric tensor fields $A_{\mu_1 \dots \mu_p}$ ($p = 0, 1, \dots, n$), for which we adopt the mathematicians' nickname "differential forms". Tensors of rank p are called p-forms.

If $\omega = (A, A_\mu, A_{\mu_1\mu_2}, \dots, A_{\mu_1\dots\mu_n})$ is a form, the generalized curl operator d is defined by

$$d\omega = \Big(0, \partial_\mu A, \partial_{\mu_1} A_{\mu_2} - \partial_{\mu_2} A_{\mu_1}, \dots, \sum_{\substack{\text{cyclic}\\ \text{permutation}}} (-1)^\pi \partial_{\mu_{\pi_1}} A_{\mu_{\pi_2}\dots\mu_{\pi_n}}\Big), \quad (1)$$

and (minus) the generalized divergence, d^+, by

$$-d^+\omega = (A^\nu, A^\nu{}_\mu, \dots, A^\nu{}_{\mu_1 \dots \mu_{n-1}}, 0)_{;\nu} . \quad (2)$$

If we introduce the scalar product

$$(\omega_A, \omega_B) = \int d^n x \sqrt{g} \sum_p \frac{1}{p!} A^{\mu_1 \dots \mu_p} B_{\mu_1 \dots \mu_p} , \quad (3)$$

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‡1 The restriction to riemannian manifolds is not essential. On a pseudo-riemannian space the KD operator D will be hermitian rather than anti-hermitian.

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then it is easy to verify that d and d^+ are mutually adjoint, as our notation suggests. In terms of d and d^+ the laplacian is [6]

$$\Delta = -dd^+ - d^+d \,. \tag{4}$$

Since $d^2 = d^{+2} = 0$ we can write this as

$$\Delta = (d - d^+)^2 \equiv D^2 \,. \tag{5}$$

D is thus a natural square root of the laplacian and we are led to suspect a connection between the KD equation [‡1]

$$(d - d^+)\,\omega = m\omega \tag{6}$$

and the Dirac equation.

To make the connection, let us specialize to an euclidean space with cartesian coordinates x_μ and introduce γ matrices as the irreducible representation of the Clifford algebra

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu} \,. \tag{7}$$

Now we can form the matrix

$$\psi^{(\omega)}_{\alpha\beta}(x) = \sum_{p=0}^{n} \frac{1}{p!}\,(\gamma^{\mu_1}\cdots\gamma^{\mu_p})_{\alpha\beta}A_{\mu_1\ldots\mu_p} \tag{8}$$

and check that

$$\gamma^\mu_{\alpha\alpha'}\partial_\mu\,\psi^{(\omega)}_{\alpha'\beta}(x) = \psi^{(D\omega)}_{\alpha\beta}(x) \,. \tag{9}$$

Each column of ψ can be thought of as a Dirac spinor so ψ (and therefore ω) represents a multiplet of degenerate fermions (β acts like an internal symmetry index).

In even dimensions the correspondence (8) is one to one only if we impose reality conditions on ψ. The irreducible $N = 2^{n/2}$ dimensional representation of the γ_μ is unique [7] and thus there exists a matrix M such that

$$\gamma^*_\mu = M\gamma_\mu M^{-1} \,. \tag{10}$$

Since ω is real ψ satisfies

$$\psi^* = M\psi M^{-1} \,. \tag{11}$$

It thus has N^2 real parameters which is the same number as in ω. The components of ω can be recovered from ψ because the antisymmetric products of γ's are linearly independent. In odd dimensions the situation is slightly more complicated. Since there are two inequivalent $N = 2^{(n-1)/2}$ dimensional irreducible representations of the γ_μ one has to use two independent ψ's.

The KD equation can be derived from a lagrangian provided the KD fields are interpreted as anticommuting variables. This establishes the correct spin-statistics relation for fermions. Using the construction of eq. (8) we find that in the massless case the lagrangian becomes tr $\psi^+\gamma^\mu\partial_\mu\psi$ in euclidean space–time. It follows that the theory is invariant under the symmetry operation

$$\psi \to S\psi U^{-1} \,, \tag{12}$$

where S is an element of O(n) (the analog of the Lorentz group in Minkowski space) and U is an element of U(N) an internal symmetry of the N-fold degeneracy. U(N) has an O(n) subgroup and the tensor character of ω reflects the transformations under the diagonal O(n) subgroup of both internal and external O(n) rotations, i.e. $\psi \to S\psi S^{-1}$.

The fermion multiplicity and mixing of space–time and internal symmetries are reminiscent of Susskind's [4] lattice fermions. Indeed the KD equation can be discretized and its continuum limit involves no doubling of the KD fields [2]. The lattice analog of p-forms are functions defined on p-dimensional hypercubes (vectors = link variables, second rank tensors = plaquette variables, etc.). Let us associate each p-form with the geometrical center of the p-cube on which it is defined. Form a new cubic lattice L from all points at which forms sit (fig. 1). L contains the lattice, the dual lattice and other points and its unit cell has volume 2^{-n} in units of the original lattice spacing. The KD field ω is just a point function $\chi(x)$ on L and if ω satisfies the massless KD equation (6) then χ satisfies the Susskind equation

$$\sum_\mu \alpha_\mu(x)[\chi(x+\mu) - \chi(x-\mu)] = 0 \,, \tag{13}$$

where

$$\alpha_\mu = \pm 1 \qquad \prod_{\text{plaquettes}} \alpha_\mu = -1 \,. \tag{14}$$

The massless case is of particular interest because of the additional chiral invariance $\psi \to \gamma_5\psi$ of the KD equation. Note that the equation possesses trivially the invariance under $\psi \to \psi\gamma_5$ in any case. In four dimensions, where $n = N = 4$, we can use these two

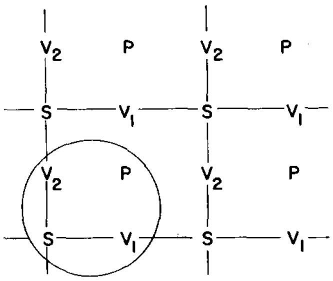

Fig. 1. The forms (antisymmetric tensors) defined on a lattice are associated with their corresponding geometrical counterparts. The figure depicts the situation in two dimensions where the scalar, vector and pseudoscalar are associated with the vertex, link and plaquette respectively. The components of the tensors can be regarded as elements associated with vertices of a new lattice of half the original spacing. The circle includes all elements of one ψ.

different transformations to characterize completely the four different solutions to the massless KD equation. Using the notation of differential forms one can show the equivalence

$$\psi \to \gamma_5 \psi \Leftrightarrow \omega \to {}^*(-1)^{p(p-1)/2}\omega \,, \tag{15a}$$

$$\psi \to \psi\gamma_5 \Leftrightarrow \omega \to {}^*(-1)^{p(p+1)/2}\omega \,. \tag{15b}$$

where * is the (Hodge) duality operator [6], (15a) anticommutes with the KD operator of eq. (5) while (15b) commutes with it. Eqs. (15) hold for any even number of dimensions and can be used on curved manifolds as well.

Before turning to curved space–time let us remark that the above discussion can be carried out also in Minkowski space although the notation is less natural because it involves a comparison of an internal $O(n)$ symmetry with an external $O(n-1, 1)$ one. Thus in the $n = 2$ dimensional example (which we used in fig. 1) we can construct

$$\psi = S + \gamma^\mu V_\mu + \mathrm{i}\gamma_5 P \tag{16}$$

in euclidean space where all γ are hermitian and all forms (S, V_μ, $\epsilon_{\mu\nu}P$) are real. In Minkowski space we find that the choice

$$\psi = \mathrm{i}S + \gamma^\mu V_\mu + \mathrm{i}\gamma_5 P \tag{17}$$

(with γ^0 and γ_5 hermitian) guarantees that the equations of motion will be real; yet in order to have the proper lagrangian

$$\mathcal{L} = \mathrm{tr}\, \psi^+ \mathrm{i}\gamma^0\gamma^\mu\partial_\mu\psi \tag{18}$$

we have to require, in the chiral basis,

$$S^* = V_0, \quad P^* = -V_1 \,. \tag{19}$$

The trace in eq. (18) guarantees the expected internal U(2) or O(2) symmetry yet the result (19), while showing the correct number of independent variables, exhibits an intricate admixture of O(1,1) tensors.

The most remarkable property of the KD equation is that is is not equivalent to a multiplet of Dirac equations in the presence of a gravitational field. Eq. (6) has the same form on an arbitrary curved manifold but eq. (8) is replaced by

$$\psi = \sum_{p=0}^{n} \frac{1}{p!}\, \gamma^{a_1} \dots \gamma^{a_p} e^{\mu_1}{}_{a_1} \dots e^{\mu_p}{}_{a_p} A_{\mu_1 \dots \mu_p} \,, \tag{20}$$

where $e^\mu{}_a$ is the local orthonormal frame $e^\mu{}_a e^\nu{}_a = g^{\mu\nu}$. This implies that ψ transforms as a sum of antisymmetric tensor representations of the local Lorentz gauge symmetry:

$$e^\mu{}_a(x) \to \Lambda_{ab}(x) e^\mu{}_b(x) \,,$$

$$\psi(x) \to S(\Lambda(x))\psi(x)S^{-1}(\Lambda(x)) \,, \tag{21}$$

rather than as a multiplet of spinors as would be required by the Lorentz invariance of the massless Dirac equation: [8]

$$\gamma^a e^\mu{}_a(\partial_\mu - \mathrm{i}\omega_\mu{}^{bc}\sigma_{bc})\psi = 0 \,. \tag{22}$$

Instead ψ satisfies

$$\gamma^a e^\mu{}_a(\partial_\mu\psi - \mathrm{i}\omega_\mu{}^{bc}[\sigma_{bc}, \psi]) = 0 \,, \tag{23}$$

which can be shown to be equivalent to (6) for $m = 0$.

This equation implies a drastic revision of many of our ideas about fermions. It predicts a violation of the equivalence principle (in the sense that the experimentally observed behavior of fermions in a rotating coordinate system is not equivalent to their behavior in a gravitational field) as well as a violation of Lorentz invariance in processes involving gravity. Concomitantly, it implies that gravitational effects can change the "internal" quantum numbers of fermions. Despite the strange nature of these processes we believe that they occur with amplitudes that are sufficiently small to have escaped experimental detection. Work is in progress to study this question in greater detail.

In curved space, the KD equation no longer decouples into four (in four dimensions) independent equations. Nonetheless the chiral transformation of eq. (15) can be defined in curved space as well and the massless KD lagrangian separates into a pair of lagrangians for self-dual and anti self-dual forms ‡2. In the flat space limit each of these lagrangians describes a pair of Weyl fermions, but in the presence of a gravitational field the two Weyl particles are coupled together. Thus the KD lagrangian forces us to a multiplicity of fermions which is not associated with an exact internal symmetry group. It is tempting to identify this multiplicity with the generation structure of the observed spectrum of quarks and leptons (but one should keep in mind that the KD equation may be relevant only on the level of preons). This temptation is strengthened by the fact that if we add internal symmetry gauge fields to the KD equation, all of the Weyl fermions of the flat space theory have the same gauge quantum numbers.

The identification of the KD multiplicity with generations has a number of immediate consequences. Firstly since the KD multiplicity is even we should expect to discover at least one more generation. Secondly generation mixing interactions would seem to be purely gravitational in origin. Local Lorentz invariance of eq. (23) seems to forbid asymmetric couplings of gauge or Higgs fields to the two Weyl fermions in a KD multiplet. This has implications for the mechanism that breaks the chiral symmetries which (we assume) prevent fermions from getting masses of $O(10^{19}$ GeV). If the symmetry breaking mechanism is soft (as in technicolor) [9] gravitational corrections to the fermion mass matrix (and thus, in our scenario, all Cabbibo–Kobayashi–Maskawa angles) will be $O(G_F^{-3/2} M_p^{-2})$. On the other hand, if the symmetry breaking is due to the vacuum expectation value of an elementary scalar field ϕ the radiative corrections are ultraviolet divergent and (assuming a cutoff at the Planck mass M_p) we expect $\delta m \sim \phi f(\ln(\phi/M_p))$. Finally, such a scenario would predict that all generation mixing effects which are not attributable to the fermion mass matrix should be suppressed by powers of the Planck mass.

The principal defect of this scenario is the lack of a general argument that guarantees that the mass matrix is Lorentz invariant. Quantum gravitational fluctuations break conventional Lorentz symmetry in the KD lagrangian. While most manifestations of this symmetry breaking are suppressed by powers of the Planck mass at low energies, the ultraviolet divergent corrections to the fermion mass matrix should be sensitive to it. Perhaps the answer to this puzzle lies in the detailed group theory of the chiral symmetry breaking mechanism, which we are far from understanding at present.

To conclude this note, we want to record our belief that the KD equation will find its most appropriate setting in the framework of Kaluza–Klein theories. Such a combination would be a completely geometric description of all fields and interactions. Note in this connection that the study of zero modes of the KD equation on compact manifolds is equivalent to the theory of harmonic forms, one of the classic problems of mathematics. A complete classification of these modes is known in terms of the topology of the manifold (de-Rham cohomology theorems) and we are guaranteed zero modes whenever some of the Betti numbers are non-zero [6]. This is in marked contrast to the Dirac equation which (for example) has no zero modes on any manifold with positive curvature [10]. Ultimately the reason for this difference is that the square of the KD operator is always the laplacian while this is only true of the Dirac operator in flat space. Thus, ironically, Dirac's hand waving derivation of his wave equation may be deeper than more sophisticated treatments based on the representation theory of the Lorentz group.

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‡2 The transformation which commutes with the KD operator, eq. (15b), splits the KD lagrangian into two on the classical level. However, by analogy with what happens to U(1) chiral symmetries in gauge theories we suspect that a combination of anomalies and topologically non-trivial quantum fluctuations of the gravitational field might cause transitions between the self dual and antiselfdual tensors.

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