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Coherent Production of Pions*

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We investigate the question of independent and coherent production of pions in high energy processes. A model for coherent production is suggested and the restrictions due to momentum conservation, charge conservation, and isospin conservation are dealt with in detail. A formalism for the isospin analysis of identical pions is developed and applied to coherent states. Results and consequences are discussed. Experimental examples are compared with the theoretical concepts and ideas.

I. INTRODUCTION

In this paper we try to test theoretically the concept of coherent pion production in high energy experiments. Let us first summarize several characteristics that are known to be common to many high energy collisions. The main products of these experiments are pions whose average multiplicity grows slowly with energy. The products of the collision move with a relatively low transverse momentum k_T (perpendicular to the incoming momentum) whose average is roughly of the order of 300–400 MeV. In the Appendix we discuss experimental examples that show these characteristics. In particular, we note that many reactions show a distribution of pions that is strongly concentrated around the centre of mass (this phenomenon is sometimes referred to as pionization). It is therefore useful to describe the situa-

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tion in terms of the c.m. momenta which we shall use throughout this paper. The relativistic phase space d^3k/ω includes a convenient enhancement factor near the centre of mass. In addition to these pions, one often observes also two leading particles. These are outgoing particles that follow the trend of the incoming momenta. They can be identical with the incoming particles but may also be resonances that are emitted with low momentum transfers from the colliding particles. The relative amount of resonance emission seems to vary from one reaction to another. One may actually be tempted to explain all data in terms of resonance production. It seems, hopeless, however, to approach the problem in such a fashion. We chose therefore a different way in the present paper—motivated by looking at the problem from a different angle, and regarding pion emission as a process that is close to "classical radiation" of a pionic field.

We will define now some key concepts that we will use. We call the emission of π_i and π_j in a reaction $A + B \to \pi_i + \pi_j + \cdots$ *uncorrelated* if

$$P(\mathbf{k}_i, \mathbf{k}_j) = P(\mathbf{k}_i)\, P(\mathbf{k}_j), \tag{1}$$

where $P(\mathbf{k}_i)$ is the probability distribution of π_i in (c.m.) momentum space. $P(\mathbf{k}_i, \mathbf{k}_j)$ is the distribution for the two pions. Experimental examples that show some evidence for such behaviour are given in the Appendix. This concept can apply to pions of either the same or different charges. We expect that to the extent that (1) is satisfied, similar uncorrelated behaviour will hold also for distributions of more particles. We define now the emission of a pion as *independent* if the same $P(\mathbf{k})$ is found in all (or many) configurations of outgoing particles where this pion is tested. We assume independence also to imply that the cross section for production of n (e.g., neutral) pions of momenta $\mathbf{k}_1 \cdots \mathbf{k}_n$ (and some other fixed configuration of particles) will be proportional to

$$\frac{d^3k_1}{2\omega_1} fP(\mathbf{k}_1) \cdots \frac{d^3k_n}{2\omega_n} fP(\mathbf{k}_n).$$

$fP(\mathbf{k})$ may in principle depend on the charge of the pion but is supposed to be fixed for a given energy of incoming particles. We learn in the Appendix that this can be sometimes regarded as a crude approximation to the actual situation. If the emission is independent, one expects to find distributions of the many-particle events similar to the Poisson distribution. A state that has such characteristics and in which also the phase of the pion wave function [whose norm is $fP(\mathbf{k})$] is fixed is the *coherent* state. In the next sections we will discuss the general characteristics of this state and explain how it can be incorporated in a T matrix description of many-particle production. Since we apply the method to pions that are emitted with low energies in c.m. it can be called *coherent pionization*. We deviate perhaps from the usual picture that this name may suggest, since we allow correlations between the pions and the leading particles. We forbid only correlations among the pions in the emitted cloud. Finally, let us introduce the concept of *identical* pions. This applies when a set of uncorrelated pions is described by the same wave function in momentum space for each pion with the exception of unique but different phases and overall magnitudes for the three different charges.

One may wonder how it is possible to talk about independent and coherent pions if there are some obvious correlations to be taken into account. The aim of this paper is to answer this question. We look for theoretical restrictions and implications with regard to these concepts. The constraints that we discuss can be divided into three general classes:

(1) four-momentum conservation (phase space);
(2) conserved quantum numbers (charge, parity, charge conjugation);
(3) isospin conservation.

Momentum conservation is taken care of by the leading particles—upon integrating on them one recovers the coherent state of pions (within our model at least). This question and related ones had already been dealt with in the literature in many different ways [1]. We develop our own version of coherent production and then contine to discuss the conserved quantum numbers and the restrictions imposed by them, a subject overlooked in many other discussions of similar models. We then develop a formalism to handle isospins of identical pions and apply it to our coherent states. The paper is divided into the following Sections:

II. — The coherent state,
III. — Emission of a coherent state,
IV. — Charge and parity,
V. — Isospin analysis of identical pions,
VI. — Isospin analysis of coherent pions,
VII. — Discussion and summary,
Appendix — Experimental examples.

We do not intend to fit experimental results or even to speculate about the exact form of the probability distribution in momentum space and its s dependence. We just study the concept of coherence to see where and when it might be applicable.

II. The Coherent State

The concept of a coherent state of bosons is quite unique in its physical interpretation and mathematical structure. This is a quantum mechanical state that is closest to a classical system in its dynamical properties [2]. A coherent state of particles can thus be regarded as a classical radiation of the corresponding field. It is used in describing electromagnetic radiation in quantum optics [2] as well as in the analysis of bremsstrahlung and the related infra-red catastrophe [3]. In the present section we apply this concept to pions. For the time being, we still disregard their quantum numbers and use only the fact that they are bosons. Modifications introduced in the following Sections can be simply implemented within the formalism of Sections II and III.

In the theory of a nonrelativistic harmonic oscillator one defines a coherent state as an eigenstate of the annihilation operator. It corresponds to a displaced ground state in configuration and momentum space and has therefore the minimal uncertainty value of $\Delta \times \Delta p$. Its time evolution looks like a classical wave packet since it retains its shape during the motion.

The definition in a relativistic theory of many bosons follows parallel lines. Let us define creation and annihilation operators satisfying

$$[a(\mathbf{k}), a^{+}(\mathbf{k}')] = \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \tag{2}$$

We define the coherent state of bosons $|f\rangle$ by the equation

$$a(\mathbf{k}) \mid f\rangle = \frac{f(k)}{\sqrt{2\omega}} \mid f\rangle, \tag{3}$$

where $f(k)$ is the momentum space wave function of each boson, k is the four-momentum, and $k_0 = \omega = \sqrt{\mathbf{k}^2 + \mu^2}$. $f(k)$ is a relativistic invariant function of k and depends on some external momenta as well. The solution to Eq. (3) is given by

$$\begin{aligned} |f\rangle &= \exp\left\{\int d^3k \left[\frac{f(k)}{\sqrt{2\omega}}\, a^{+}(\mathbf{k}) - \frac{1}{2}\,\frac{|f(k)|^2}{2\omega}\right]\right\} \mid 0\rangle \\ &= \exp\left\{\int d^3k \left[\frac{f(k)}{\sqrt{2\omega}}\, a^{+}(\mathbf{k}) - \frac{f^*(k)}{\sqrt{2\omega}}\, a(\mathbf{k})\right]\right\} \mid 0\rangle \equiv S(f) \mid 0\rangle, \end{aligned} \tag{4}$$

where we normalized $|f\rangle$ to $\langle f|f\rangle = 1$. The basic Eq. (3) gives the clue to the classical behaviour: The expectation value of a second quantized boson field within the state $|f\rangle$ will be given by the classical field with momentum distribution $f(k)$.

The expectation value of the four momentum operator P_μ is

$$\langle f|\, P_\mu \,|f\rangle = \langle f| \int d^3k\, k_\mu a^{+}(\mathbf{k})\, a(\mathbf{k}) \,|f\rangle = \int d^3k\, k_\mu \frac{|f(k)|^2}{2\omega}. \tag{5}$$

The expectation value of the number operator N is defined as $\bar{n}$ and given by

$$\langle f|\, N \,|f\rangle \equiv \bar{n} = \langle f| \int d^3k\, a^{+}(\mathbf{k})\, a(\mathbf{k}) \,|f\rangle = \int d^3k \frac{|f(k)|^2}{2\omega}. \tag{6}$$

Obviously, $|f\rangle$ is a combination of all n-particle states. A straightforward calculation leads to

$$|f\rangle = \sum_n \left(e^{-\bar{n}} \frac{\bar{n}^n}{n!}\right)^{1/2} |n\rangle, \tag{7}$$

thus exhibiting a Poisson distribution in the states $|n\rangle$.

In dealing with the production of a coherent state, we have to project out of it the piece that corresponds to a definite four-momentum K. We will denote this new state by $|f, K\rangle$. It is given by

$$\begin{aligned} |f, K\rangle &= \frac{1}{(2\pi)^4} \int d^4x \, e^{-iK\cdot x} S(fe^{ik\cdot x}) \, |0\rangle \\ &= \frac{1}{(2\pi)^4} \int d^4x \, e^{-iK\cdot x} \exp\left\{-\frac{\bar{n}}{2} + \int d^3k \, \frac{f(k)}{\sqrt{2\omega}} \, a^+(k) \, e^{ik\cdot x}\right\} |0\rangle \end{aligned} \tag{8}$$

and obeys

$$\int d^4K \, |f, K\rangle = |f\rangle, \tag{9}$$

$$\langle f \,|\, f, K\rangle \equiv \rho_f(K) = \frac{1}{(2\pi)^4} \int d^4x \, e^{-iK\cdot x} \exp\left\{\int \frac{d^3k}{2\omega} \, |f|^2 \, (e^{ik\cdot x} - 1)\right\}, \tag{10}$$

$$\langle f, K' \,|\, f, K\rangle = \delta^{(4)}(K' - K) \, \rho_f(K). \tag{11}$$

It is instructive to decompose the quantity $\rho_f(K)$ in a series of the various n-particle contributions. This is achieved by a decomposition of the integrand in Eq. (10) in powers of e^{ikx} :

$$\begin{aligned} \rho_f(K) &= \sum_{n=0}^{\infty} \rho_f^{(n)}(K) \\ &= e^{-\bar{n}} \Big\{\delta^{(4)}(K) + |f(K)|^2 \, \delta(K^2 - \mu^2) \, \theta(K_0) + \frac{1}{2!} \int d^4k \, |f(k)|^2 \, |f(K-k)|^2 \\ &\quad \times \delta(k^2 - \mu^2) \, \delta[(K-k)^2 - \mu^2] \, \theta(k_0) \, \theta(K_0 - k_0) + \cdots\Big\}. \end{aligned} \tag{12}$$

Equation (12) reveals the spectrum structure that one would expect: a contribution at $K_\mu = 0$ from the vacuum component, one at $K^2 = \mu^2$ from the one-particle state and a continuum that starts from the threshold of two particles.

For the sake of further use in the next section, we list also some properties of the scalar products of two different coherent states:

$$\langle g \mid f\rangle = \exp\left\{-\frac{1}{2}\int \frac{d^3k}{2\omega}\left(|f|^2 + |g|^2 - 2g^* f\right)\right\}. \tag{13}$$

Equation (13) shows that two different coherent states are not orthogonal to each other (they are not eigenstates of Hermitian operators). Nevertheless, they do form an over-complete set [2]. The analog of Eq. (10) is

$$\begin{aligned}\langle g \mid f, K\rangle &\equiv \rho_{g,f}(K)\\ &= \frac{1}{(2\pi)^4}\int d^4x e^{-iK\cdot x}\exp\left\{-\frac{1}{2}\int\frac{d^3k}{2\omega}\left(|f|^2 + |g|^2 - 2g^* f e^{ik\cdot x}\right)\right\}. \end{aligned}\tag{14}$$

The calculation of quantities like $\rho_f(K)$ or $\rho_f^{(n)}(K)$ is not an easy matter. Thus $\rho_f^{(n)}(K)$ can be rewritten as

$$\rho_f^{(n)}(K) = \frac{e^{-\bar{n}}}{n!}\int \frac{d^3k_1}{2\omega_1}\cdots\frac{d^3k_n}{2\omega_n}\,|f(k_1)|^2\cdots|f(k_n)|^2\,\delta^{(4)}(k_1 + \cdots + k_n - K). \tag{15}$$

To simplify matters, we can define normalized distributions $\tilde{\rho}_f^{(n)}(K)$ such that

$$\rho_f^{(n)}(K) = e^{-\bar{n}}\frac{\bar{n}^n}{n!}\tilde{\rho}_f^{(n)}(K), \qquad \int d^4K\,\tilde{\rho}_f^{(n)}(K) = 1. \tag{16}$$

One can then use the central limit theorem and find that

$$\tilde{\rho}_f^{(n)}(K) \approx \frac{1}{n^2}\frac{\sqrt{\det\eta}}{4\pi^2}\exp\left\{-\frac{1}{2n}\eta_{\mu\nu}(K^\mu - n\bar{k}^\mu)(K^\nu - n\bar{k}^\nu)\right\}, \tag{17}$$

where we used

$$\begin{aligned}\bar{k}^\mu &= \frac{1}{\bar{n}}\int\frac{d^3k}{2\omega}k^\mu\,|f(k)|^2,\\ \frac{\eta_{\mu\nu}}{\bar{n}}&\int(k^\nu - \bar{k}^\nu)(k^\sigma - \bar{k}^\sigma)\,|f(k)|^2\frac{d^3k}{2\omega} = \delta_\mu{}^\sigma.\end{aligned}\tag{18}$$

This result was given by Van Hove [4] and similar expressions were analyzed in detail by Lurçat and Mazur [5]. Let us discuss here briefly the expected form for $\tilde{\rho}_f^{(n)}$ if $f(k)$ has the characteristics of the distribution functions of the pions described in the Appendix. A reasonable guess would be $\bar{k}^\mu = (\bar{\omega}; \mathbf{0})$, with $\eta_{\mu\nu}$ a diagonal matrix with elements $(\sigma_E^{-2}, \sigma_T^{-2}, \sigma_T^{-2}, \sigma_L^{-2})$, where T and L designate transverse and longitudinal directions, respectively. There is obviously a connection between these terms given by $\sigma_E^2 = 2\sigma_T^2 + \sigma_L^2 + \mu^2 - \bar{\omega}^2$. It then follows from (17) that

$$\tilde{\rho}_f^{(n)}(K) \approx \frac{1}{n^2} \frac{1}{4\pi^2\sigma_T^2\sigma_E\sigma_L} \exp\left\{-\frac{1}{n}\left[\frac{(K_0 - n\bar{\omega})^2}{2\sigma_E^2} + \frac{K_T^2}{2\sigma_T^2} + \frac{K_L^2}{2\sigma_L^2}\right]\right\}. \quad (19)$$

Equation (19) tells us that the over-all distribution of the coherent pions is peaked around a linearly increasing energy with an increasing width as expected from a typical random walk problem.

III. Emission of a Coherent State

In this section we discuss a formalism that describes a process in which two incoming particles (with momenta q_1 and q_2) produce two outgoing leading particles (with momenta p_1 and p_2) and n mesons of momenta $k_1 \cdots k_n$ which are part of a coherent state. For the moment we continue to ignore the quantum numbers of the pions (we will discuss this problem in the next section). We propose now the following factorized S matrix structure

$$\langle p_1 p_2 k_1 \cdots k_n \mid S \mid q_1 q_2 \rangle$$
$$= i \int d^4x\, e^{ix\cdot(p_1+p_2-q_1-q_2)} \langle p_1 p_2 k_1 \cdots k_n \mid S(fe^{ikx})\, \tilde{T} \mid q_1 q_2 \rangle. \quad (20)$$

To the extent that the incoming particles are not mesons of the kind appearing in the coherent cloud [or, if they are such mesons, they have momenta outside the range of $f(k)$] equation (20) can be brought into a completely factorized form

$$\langle p_1 p_2 k_1 \cdots k_n \mid S \mid q_1 q_2 \rangle$$
$$= i \int d^4K\, (2\pi)^4\, \delta^{(4)}(K + p_1 + p_2 - q_1 - q_2) \langle k_1 \cdots k_n \mid f, K \rangle \langle p_1 p_2 \mid \tilde{T} \mid q_1 q_2 \rangle. \quad (21)$$

$\tilde{T}$ acts only on the particles $q_1 q_2 p_1 p_2$ that form what we call the "skeleton" of the process. It can thus depend on the invariant variables:

$$\begin{aligned} s &= (q_1 + q_2)^2, & \bar{s} &= (p_1 + p_2)^2, \\ t &= (q_1 - p_1)^2, & \bar{t} &= (q_2 - p_2)^2, \\ u &= (q_1 - p_2)^2, & \bar{u} &= (q_2 - p_1)^2. \end{aligned} \quad (22)$$

The sum of all these variables is given by

$$s + t + u + \bar{s} + \bar{t} + \bar{u} = K^2 + 2\Sigma, \quad (23)$$

where

$$K = q_1 + q_2 - p_1 - p_2, \qquad \Sigma = q_1^2 + q_2^2 + p_1^2 + p_2^2.$$

Note that $f(k)$ is an invariant function of k and depends therefore on the four momenta of the skeleton. We refer to this fact by using the notation $f_{pq}(k)$. It assures the invariance of the expressions (20) and (21) under the Poincaré group.

The form (21) leads to the following result for the cross section of n meson production

$$\sigma_{2+n} = \int (dp)\, \rho^{(n)}_{f_{pq}}(q_1 + q_2 - p_1 - p_2)\, |\langle p \mid \tilde{T} \mid q \rangle|^2, \tag{24}$$

where (dp) stands for the invariant phase space element of the outgoing leading particles and the relevant flux factor. $\langle p \mid \tilde{T} \mid q \rangle$ is an abbreviation for the skeleton matrix element. Equation (24) is formally similar to the two-particle production cross section

$$\sigma_2 = \int (dp)\, \delta^{(4)}(q_1 + q_2 - p_1 - p_2)\, |\langle p \mid T \mid q \rangle|^2, \tag{25}$$

with the $\rho^{(n)}$ replacing the δ function. In other words, $\rho^{(n)}$ describes the distribution of four-momenta absorbed in the mesonic cloud. We leave it yet as an open question whether the recipe (24) can be smoothly continued to $n = 0$ to give $\sigma_2 = \tilde{\sigma}$, where

$$\begin{aligned} \tilde{\sigma} &= \int (dp)\, \rho^{(0)}_{f_{pq}}(q_1 + q_2 - p_1 - p_2)\, |\langle p \mid \tilde{T} \mid q \rangle|^2 \\ &= \int (dp)\, \delta^{(4)}(q_1 + q_2 - p_1 - p_2)\, e^{-\bar{n}_{pq}}\, |\langle p \mid \tilde{T} \mid q \rangle|^2. \end{aligned} \tag{26}$$

Equation (19) gave us the approximate form of $\tilde{\rho}^{(n)}$ which turned out to be concentrated around $K_0 = n\bar{\omega}$, $\mathbf{K} = 0$. If we assume that $\langle p \mid \tilde{T} \mid q \rangle$ is independent of K^2, then, at least until the end of phase space is reached, we can approximate (24) by

$$\begin{aligned} \sigma_{2+n} &= \int (dp)\, d^4K\, \rho^{(n)}_{f_{pq}}(K)\, \delta^{(4)}(K + p_1 + p_2 - q_1 - q_2)\, |\langle p \mid \tilde{T} \mid q \rangle|^2 \\ &\approx \int (dp)\, \delta^{(4)}(p + \bar{K} - q)\, |\langle p \mid \tilde{T} \mid q \rangle|^2 \int d^4K\, \rho^{(n)}_{f_{pq}}(K) \\ &\approx \int (dp)\, \delta^{(4)}(p_1 + p_2 - q_1 - q_2)\, |\langle p \mid \tilde{T} \mid q \rangle|^2\, e^{-\bar{n}_{pq}} \frac{\bar{n}_{pq}^n}{n!}, \end{aligned} \tag{27}$$

which means a Poisson distribution for the differential cross section. The last equality implies a structure in $\tilde{T}$ to which we will return below. Further, if $\bar{n}$ depends on q only then we have

$$\sigma_{2+n} = \frac{\bar{n}^n}{n!}\,\tilde{\sigma}. \tag{28}$$

This calculation makes sense only provided phase-space restrictions can be avoided. In other words, if the number of pions is smaller than the maximum allowed by energy conservation, then

$$n\bar{\omega} < \sqrt{s} - m_1 - m_2\,, \tag{29}$$

where m_1 and m_2 are the masses of p_1 and p_2. This works best for an $f(k)$ that is concentrated around the c.m. with a narrow width. For high n that violate the inequality (29) we have to expect distortions of the distribution law (28).

The skeleton can be either elastic or inelastic. By elastic skeleton we mean that the outgoing particles in the skeleton are the same as the incoming one. This does not imply $\sigma_2 = \tilde{\sigma}$; to this question we return in a minute. An inelastic skeleton can have resonances among its outgoing particles. Actually, a skeleton does not have to be a four-particle object and can also consist out of three particles or five, etc. The existence of two leading particles is, of course, suggestive of a four-point skeleton. In the case of an elastic skeleton we have some evidence that Eq. (28) cannot be continued to $n = 0$. This is pointed out in the Appendix. We have, therefore, to rely on unitarity to give us the elastic part $\langle p \mid T \mid q\rangle$ in terms of all the inelastic reactions. As a crude approximation, one may consider a model in which all inelastic reactions are described by Eq. (21) with an elastic skeleton. Unitarity leads then to the condition

$$\begin{aligned} &i[\langle p_1p_2 \mid T^+ \mid q_1q_2\rangle - \langle p_1p_2 \mid T \mid q_1q_2\rangle] \\ &\qquad - \int \frac{d^3p_1{}'}{2E_1{}'}\,\frac{d^3p_2{}'}{2E_2{}'}\,(2\pi)^4\,\delta^{(4)}(p_1{}' + p_2{}' - q_1 - q_2)\langle p \mid T^+ \mid p'\rangle\langle p' \mid T \mid q\rangle \\ &= \sum_{n=1}^{\infty} \int \frac{d^3p_1{}'}{2E_1{}'}\,\frac{d^3p_2{}'}{2E_2{}'}\,(2\pi)^4\,\rho^{(n)}_{f_{p'p},f_{p'q}}(q_1 + q_2 - p_1 - p_2) \\ &\qquad \times \langle p_1p_2 \mid \tilde{T}^+ \mid p_1{}'p_2{}'\rangle\langle p_1{}'p_2{}' \mid \tilde{T} \mid q_1q_2\rangle, \end{aligned} \tag{30}$$

where $\rho^{(n)}_{f_{p'p},f_{p'q}}$ are the n-particle contributions to an object of the type defined in Eq. (14). The right side of Eq. (30) is analogous to Van Hove's "overlap integral" [4] that determines the t structure of the elastic amplitude. Note that an explicit t structure can and should exist in $\langle p \mid \tilde{T} \mid q\rangle$ [6] as we shall see below.

It is interesting to see how the bremsstrahlung theory [3] which actually suggests this formalism, solves the unitarity problem. The function f_{pq} is given in this case by

$$(2\pi)^{3/2} f_{pq}(k) = e_1' \frac{\epsilon \cdot p_1}{k \cdot p_1} + e_2' \frac{\epsilon \cdot p_2}{k \cdot p_2} - e_1 \frac{\epsilon \cdot q_1}{k \cdot p_1} - e_2 \frac{\epsilon \cdot q_2}{k \cdot p_2}, \tag{31}$$

where e_i are the various charges and ϵ is the photon's polarization vector. Clearly, f is peaked around $k = 0$ and the whole treatment is verified by QED only in the limit $k \to 0$ [3]. Then it turns out that indeed

$$\langle p \mid T \mid q \rangle = \langle p \mid \tilde{T} \mid q \rangle \, e^{-\bar{n}_{pq}/2}, \tag{32}$$

and unitarity is satisfied, provided $\tilde{T}$ satisfies elastic unitarity. To see this, we rewrite (30) with the use of (32):

$$i[\langle p \mid T^{+} \mid q \rangle - \langle p \mid T \mid q \rangle]$$
$$= \int \frac{d^3p_1'}{2E_1'} \frac{d^3p_2'}{2E_2'} \langle p \mid \tilde{T}^{+} \mid p' \rangle \langle p' \mid \tilde{T} \mid q \rangle (2\pi)^4 \int d^4x$$
$$\times \exp \left\{ -\frac{1}{2} \int \frac{d^3k}{2\omega} \left(|f_{p'p}|^2 + |f_{p'q}|^2 - 2 f^{*}_{p'p} e^{ikx} f_{p'q} \right) \right\} e^{ix(p_1' + p_2' - q_1 - q_2)}.$$

Using the properties of (31) and replacing here the e^{ikx} in the integrand by 1 (the limit $k \to 0$!), we find

$$i[\langle p \mid T^{+} \mid q \rangle - \langle p \mid T \mid q \rangle]$$
$$= e^{-\bar{n}_{qp}/2} \int \frac{d^3p_1' \, d^3p_2'}{2E_1' \, 2E_2'} \langle p \mid \tilde{T}^{+} \mid p' \rangle \langle p' \mid \tilde{T} \mid q \rangle (2\pi)^4 \, \delta^{(4)}(p_1' + p_2' - q_1 - q_2), \tag{33}$$

which shows that the ansatz (32) works provided $\langle p \mid \tilde{T} \mid q \rangle$ obeys by itself a unitarity equation.

There clearly are several important differences between the formalism of bremsstrahlung and the emission of mesons in high energy collision. The first is that the identification (32) cannot be made in our case. Another is that the limit $k \to 0$ is not justifiable and cannot be obtained with massive (and energetic) mesons. That can be circumvented by having a skeleton matrix element that does not vary significantly with K. A very important third difference is that we may choose f to depend on q only. In the Appendix we show characteristic distributions that depend only on k_T and k_L. These variables can be given an invariant definition in terms of $k \cdot q_1$, $k \cdot q_2$, $q_1 \cdot q_2$ and the masses involved. Therefore one needs no p dependence. This then makes it possible to go from Eq. (27) to Eq. (28) and get simple relations for the integrated cross sections.

Let us now investigate shortly some of the properties of the skeleton in the high-energy collisions. To be specific, we will usually think of an elastic skeleton although other types can also be constructed. We already hinted after Eq. (27) that it has to have some structure. Otherwise, phase space effects of the leading particles should be felt. We learn from experiment that the leading particles are indeed confined to low transverse momenta, i.e., also low momentum transfers. Such a dependence can be incorporated in $\langle p \mid \tilde{T} \mid q \rangle$. We note that an explicit t dependence will be effectively weakened for higher multiplicities. That can be seen by looking at the relevant differential phase space element

$$\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\,d^4K\,\delta^{(4)}(p_1 + p_2 + K - q_1 - q_2) \to \frac{d\varphi\; dt\; d^4K}{8q\sqrt{s}\,R}, \tag{34}$$

where φ is the azimuthal angle and q is the value of the incoming momenta $\mathbf{q}_1$ and $\mathbf{q}_2$ in the c.m. R is given by

$$R = 1 - \frac{K_0}{\sqrt{s}} + \frac{E_1 K \cos\theta_{p_1 K}}{p\sqrt{s}} + \frac{E_1 K}{\sqrt{s}}\,\frac{\partial \cos\theta_{p_1 K}}{\partial \cos\theta_{p_1 q_1}}\,\frac{\partial \cos\theta_{p_1 q_1}}{\partial p}, \tag{35}$$

where K and p are the magnitudes of $\mathbf{K}$ and $\mathbf{p}_1$ and $\theta_{p_1 K}$ is the angle between them. For low values of K_0 (low multiplicities), $R \to 1$. Hence a strong t-dependent effect introduced in $\tilde{T}$ will show up for low n and get distorted for high n. This may be quite consistent with observed trends. We will not investigate further this question since in this paper we are mainly concerned with the consistency of the whole picture rather than the fine details.

The explicit construction of an example of coherent production of mesons shows that independent productions can take place. Coherence is also a statement about the phases that are not directly measurable. They will, however, be important when we discuss the isospin question in Section VI. The easiest things to measure are of course the cross sections. Their distribution, suggested by Eq. (28), will get modified by considerations of the quantum numbers of the pions to which we turn in the next Section.

IV. Charge and Parity

The coherent state must have a fixed electric charge that matches the charge of the skeleton. This is not true of simple charged coherent states of the types ($i = +$ or $-$)

$$|f_i\rangle = \exp\left\{-\frac{1}{2}\int d^3k\,\frac{|f_i|^2}{2\omega} + \int d^3k\,\frac{f_i(k)}{\sqrt{2\omega}}\,a_i^{+}(k)\right\} |0\rangle \qquad i = +, 0, -. \tag{36}$$

One way to deal with the problem can be to start from the state

$$|F\rangle = |f_+\rangle\,|f_0\rangle\,|f_-\rangle \tag{37}$$

and project out the required charge. An alternative is to define a state $|f_+f_-, Q\rangle$ obeying the equation

$$a_+(\mathbf{k})\,a_-(\mathbf{k})\,|f_+f_-, Q\rangle = \frac{1}{2\omega} f_+(k) f_-(k)\,|f_+f_-, Q\rangle \tag{38}$$

which has a definite charge Q. This is an analog of Eq. (3) and can serve as a definition of a coherent state of charged particles. The solution to Eq. (38) is

$$|f_+f_-, Q\rangle = C^{-1/2} \sum_n \frac{1}{(n+Q)!\,n!} \left(\int \frac{d^3k}{\sqrt{2\omega}} f_+(k)\,a_+{}^+(\mathbf{k})\right)^{n+Q} \times \left(\int \frac{d^3k}{\sqrt{2\omega}} f_-(k)\,a_-{}^+(\mathbf{k})\right)^{n} |0\rangle, \tag{39}$$

where the sum starts from $n = 0$ for positive Q and from $n = -Q$ for negative Q. The normalization constant C turns out to be

$$C = (-i)^Q J_Q(2ix), \tag{40}$$

where

$$x^2 = \int \frac{d^3k}{2\omega} |f_+(k)|^2 \int \frac{d^3k}{2\omega} |f_-(k)|^2. \tag{41}$$

This parameter is equal to the average value of n^2 in the resulting distribution of $(n+Q)$ positive and n negative charged states

$$x^2 = \langle n^2 \rangle. \tag{42}$$

It is straightforward to show that the projection of $|F\rangle$ onto a specific charge Q does indeed contain this state. It is

$$|f, Q\rangle = |f_0\rangle |f_+f_-, Q\rangle \tag{43}$$

which we will regard as the right choice to take the place of $|f\rangle$ in Eq. (21). The distribution of charged particles that results from this state was discussed by us in Ref. [7]. It is the conditional distribution reached by multiplying two Poisson distributions (for the two different charges) and projecting out the right total charge. Thus for a cloud of charge Q the cross section for $n + Q$ positive and n negative pions is given by

$$P_n^{(Q)} = P_{n+Q}^{(-Q)} = \frac{i^Q}{J_Q(2ix)} \frac{x^{2n+Q}}{n!(n+Q)!}, \tag{44}$$

and the average is

$$\langle n \rangle = -ix \frac{J_{Q+1}(2ix)}{J_Q(2ix)}. \tag{45}$$

We will return to a discussion of the distribution in Section VII.

We come now to the question of parity conservation. If the skeleton contains two spinors among its four particles, this question can be overcome by inserting $\gamma_5 f$ instead of f in the relevant expressions. Thus (20) becomes

$$\langle p_1 p_2 k_1 \cdots k_n \mid S \mid q_1 q_2 \rangle$$
$$= i \int d^4x \, e^{ix(p_1+p_2-q_1-q_2)} \langle p_1 p_2 k_1 \cdots k_n \mid S(\gamma_5 f e^{ikx}) \, \tilde{T} \mid q_1 q_2 \rangle, \tag{46}$$

and this expression is invariant under parity. This little change has some severe consequences — it leads to a distinction between even and odd numbers of emitted pions. Whereas the even numbers can still be described by the arguments in Section II which lead to the basic $\tilde{\sigma}$, the odd number of pions will be described in terms of

$$\tilde{\sigma}' = \int (dp) \, e^{-\bar{n}_{pq}} \, \delta^{(4)}(p_1 + p_2 - q_1 - q_2) \, |\langle p \mid \gamma_5 \tilde{T} \mid q \rangle|^2. \tag{47}$$

The numerical differences between $\tilde{\sigma}$ and $\tilde{\sigma}'$ can of course affect the distribution. We might, however, sometimes find small deviations that will not be detected on the logarithmic scale. For instance, in the limiting case in which the incoming baryon is very fast and the outgoing very slow, one has $|\bar{u}(p_1)\, u(q_1)| \approx |\bar{u}(p_1)\, \gamma_5 u(q_1)|$.

Another way to realize the situation is to consider the case of symmetric pionic distributions, $f(\mathbf{k}) = f(-\mathbf{k})$. Then it is clear that the allowed transitions in the skeleton have to be from even j to even j and from odd j to odd j. However, in the case of an even number of pions we have also even (odd) $l \to$ even (odd) l, whereas if the number of emitted pions is odd we find even (odd) $l \to$ odd (even) l. Hence a spin transition must be involved. All this means that the completely factorized form (21) cannot be obtained and, in the best case, it breaks down into somewhat different distributions for even and odd pions.

In practice, it turns out that this problem often does not play an important role. Thus if one looks at the distribution of charged pions one sums over the neutral ones. The neutral pions are then included in an effective skeleton and the distribution of the charged ones is given by $| \gamma_5 f_+ \gamma_5 f_-, Q\rangle$ which has always either odd or even numbers of γ_5 matrices. One way to completely avoid the parity problem is simply to discuss only emission of scalar pairs of pions. This is of course farther away from independent pion production. We will return and discuss such an approach in Sections VI and VII. In future, we will refrain from using explicitly the γ_5 factor although it will be implicitly assumed.

One immediate important consequence of the parity problem is that a skeleton of spinless particles cannot emit a coherent state $| f, Q\rangle$ of pions. Moreover, one may expect that the more spins there are in the skeleton the easier it is to connect it to a pionic coherent state. It is interesting to note in this connection that $\sigma_T(\pi\pi) < \sigma_T(\pi p) < \sigma_T(pp)$ and the direction of the inequalities is also that of the number of spins involved. If independent pion production is an important inelastic mode, then this correlation may be meaningful.

Several selection rules arise for neutral systems from charge conjugation considerations. Thus a skeleton of four pions can be connected only to even number of pions, which is the same condition as that of parity conservation. A neutral system of identical pions has positive charge conjugation. Thus it cannot couple, e.g., to e^+e^- (via a photon). Similarly $\bar{p}p$ annihilation at rest is restricted by charge conjugation. Both e^+e^- and $\bar{p}p$ are different from πp and pp collisions. e^+e^- produces hadronic matter via a photon and it is unclear what can serve as the skeleton. $\bar{p}p$ collisions, although being pure hadronic reactions, have a small elastic skeleton. Hence once again it is not clear how to implement our model in this case. The model can hopefully be used in πp and pp reactions in which charge conjugation does not impose additional restrictions.

V. Isospin Analysis of Identical Pions

We develop here a formalism that enables us to deal with the isospin analysis of a system of identical pions. We limit ourselves to identical pions since this case renders itself to an elegant and simple treatment. We will give an argument at the end of this Section showing that this limitation does not actually prevent us from reaching the optimal situation (lowest isospin) in coherent states.

We start by defining a normalized momentum space distribution $\varphi(k)$ satisfying

$$\int \frac{d^3k}{2\omega} \, | \varphi(k)|^2 = 1. \tag{48}$$

The fact that the pions are identical is summarized in the assumption

$$f_i(k) = f_i \varphi(k), \qquad i = +, 0, -, \tag{49}$$

where the f_i are three constants. The magnitudes and phases of the f_i determine the isospin structure of a definite combination of identical pions.

Let us now define three operators

$$a_i^+ = \int d^3k \, \frac{\varphi(k)}{\sqrt{2\omega}} \, a_i^+(\mathbf{k}), \tag{50}$$

which obey the commutation relations

$$[a_i \, , a_j^+] = \delta_{ij} \, . \tag{51}$$

The isospin generators for the system of identical pions can be simply expressed in terms of these operators. They are

$$\mathbf{I} = a_i^+ \boldsymbol{\tau}_{ij} a_j \, , \tag{52}$$

where

$$\tau_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \tau_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \tau_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{53}$$

Let us now define the number operators

$$N_i = a_i^+ a_i \, , \qquad N = N_+ + N_0 + N_- \, , \tag{54}$$

and the bilinear isoscalar creation and annihilation operators

$$A = a_0 a_0 - 2a_+ a_- \, , \qquad [\mathbf{I}, A] = [\mathbf{I}, A^+] = 0. \tag{55}$$

The three operators N, A and A^+ play a key role in the isospin analysis. It is interesting to note that they close on an algebra [8]

$$[N, A] = -2A, \qquad [N, A^+] = 2A^+, \qquad [A, A^+] = 4N + 6. \tag{56}$$

Their importance stems from the fact that the operator $\mathbf{I}^2$ can be written in terms of them as

$$\mathbf{I}^2 = N(N+1) - A^+ A \tag{57}$$

as can be seen by a straightforward calculation. It follows from this equation that a state of n-identical pions will have isospin $I = n$ if and only if

$$A \mid I = n, n\rangle = 0. \tag{58}$$

It is simple to construct such a state by using the following operator (note the mixed isospin property)

$$T^{+} = a_{+}^{+} + \sqrt{2}\, a_{0}^{+} + a_{-}^{+}. \tag{59}$$

This operator obeys

$$\tfrac{1}{2}[A, T^{+}] = -a_{+} + \sqrt{2}\, a_{0} - a_{-} \equiv U \qquad [U, T^{+}] = 0 \tag{60}$$

Because of these properties it is evident that

$$A(T^{+})^{n} \mid 0\rangle = 0; \tag{61}$$

hence every I_z projection of the state $(T^{+})^{n} \mid 0\rangle$ has an isospin of $I = n$. Actually, this state contains all $n + 1$ I_z projections. We can write them out explicitly and define through them the states

$$\mid I = n, I_z, n\rangle = B^{-1/2} \sum_{p} \frac{n!}{(I_z + p)!\, p!\,(n - 2p - I_z)!} (a_{+}^{+})^{I_z+p} (\sqrt{2}\, a_{0}^{+})^{n-2p-I_z} (a_{-}^{+})^{p} \mid 0\rangle, \tag{62}$$

where the sum is over all integer p such that the factorials can be defined. B is a normalization constant equal to

$$B = \sum_{p} \frac{(n!)^{2}\, 2^{n-2p-I_z}}{(I_z + p)!\, p!\,(n - 2p - I_z)!}\,. \tag{63}$$

A system of n identical pions can include, in addition to $I = n$, also all isospins of $n - 2$, $n - 4$,... down to 0 or 1. Altogether, these form $\frac{1}{2}(n + 1)(n + 2)$ states, characteristic of the completely symmetric combination. We can prove that this is the case by direct construction of the isospin states. We have already seen that A^{+} is a creation operator of an $I = 0$ system; indeed,

$$\mid I = 0, n = 2m\rangle = \frac{1}{\sqrt{(2m + 1)!}} (A^{+})^{m} \mid 0\rangle. \tag{64}$$

Thus we have created the lowest isospin for even number of pions. It is now simple to construct the general state in terms of (62)

$$\mid I, I_z, n = 2m + I\rangle = D^{-1/2}(A^{+})^{m} \mid I, I_z, n = I\rangle. \tag{65}$$

D is a normalization constant equal to

$$D = \frac{4^m m! \, \Gamma(m + I + \frac{3}{2})}{\Gamma(I + \frac{3}{2})}. \tag{66}$$

The fact that all the different states (65) form an orthonormal system can be shown by using the property (60). Simple calculation of the number of states shows that we constructed in this way all possible isospin states of identical pions.

Let us apply this formalism to the coherent state $| F\rangle$ of Eq. (37). It is defined now in terms of identical pions and satisfies

$$a_i \,| F\rangle = f_i \,| F\rangle, \qquad i = +, 0, -. \tag{67}$$

In particular,

$$A \,| F\rangle = (f_0^2 - 2f_+f_-) | F\rangle. \tag{68}$$

We see that if we choose $f_0^2 = 2f_+f_-$, we have a coherent state which contains only states with $I = n$. In other words, this choice of the parameters leads to the maximal isospin content. In our application to physics, we try to achieve the opposite goal, viz., to minimize the isospins of the coherent state since we are bound by the isospins of the skeleton. We find by using Eq. (57) that

$$\langle \mathbf{I}^2 \rangle = \langle F | \mathbf{I}^2 | F\rangle = \bar{n}(\bar{n} + 2) - | f_0^2 - 2f_+f_- |^2, \tag{69}$$

where

$$\bar{n} \equiv \langle F | N | F\rangle = | f_+ |^2 + | f_0 |^2 + | f_- |^2. \tag{70}$$

It follows from (69) that $\langle \mathbf{I}^2 \rangle$ obtains its minimal value if

$$\arg(f_0^2) = \pi + \arg(f_+f_-) \qquad \text{and} \qquad | f_+ | = | f_- | . \tag{71}$$

These are also the conditions that ensure that the state $| F\rangle$ has no preferred direction in isospace, $\langle F | \mathbf{I} | F\rangle = 0$. Thus the minimal value is reached by random walk in isospace

$$\langle \mathbf{I}^2 \rangle_{\min} = 2\bar{n} \tag{72}$$

We can show quite generaly that this situation cannot be improved if we consider a coherent state $| F\rangle$ of nonidentical pions. Indeed, in this case direct computation leads to [the use of Eq. (52) is now forbidden!]

$$\langle \mathbf{I}^2 \rangle = 2\bar{n} + \left[\int \frac{d^3k}{2\omega} \left(| f_+(k)|^2 - | f_-(k)|^2\right)\right]^2 + 2 \left| \int \frac{d^3k}{2\omega} \left[f_+^*(k) f_0(k) + f_0^*(k) f_-(k)\right]\right|^2. \tag{73}$$

The minimal value is once again given by Eq. (72). It is reached when the two squared terms vanish and in the limit of identical pions these conditions reduce to (71). We expect a similar situation to prevail also when we deal with the more physical state $|f, Q\rangle$ in the next section. We think, therefore, that the restriction to identical pions does not prevent us from reaching the minimal possible value of the isospin average of a coherent state.

VI. Isospin Analysis of Coherent Pions

Let us now apply the methods developed in the previous section to the coherent state of Eq. (43), namely, $|f, Q\rangle = |f_0\rangle\, |f_+f_-, Q\rangle$. We will limit ourselves in the numerical calculations to the case $Q = 0$. Other Q values can be treated similarly. The main idea is that the model (20) should not violate isospin conservation. That cannot be achieved in an absolute fashion with the emission of $|f, Q\rangle$ since this state contains all possible isospins. We may still hope that it does make sense as an approximation. That will be the case if the main isospin content of $|f, Q\rangle$ can be matched by the skeleton. Thus an elastic skeleton of πp can carry up to $I = 3$, and an elastic skeleton of pp up to $I = 2$. Hence the imposition of the necessary isospin cuts on $|f, Q = 0\rangle$ will make small difference only if its isospins are confined mainly to the above-mentioned regions.

The result of Eq. (72) may look quite pessimistic. Nevertheless, one should remember that $|F\rangle$ contains all the possible I_z projections. By limiting ourselves to $|f, Q = 0\rangle$ we can hope to do better. The calculation in this case is much more difficult than that for $|F\rangle$. The reason is that $|f, Q = 0\rangle$ is no longer an eigenstate of a_+ and a_-, separately. It is, however, an eigenstate of A:

$$A\,|f, Q\rangle = (f_0^2 - 2f_+f_-)|f, Q\rangle, \qquad a_0\,|f, Q\rangle = f_0\,|f, Q\rangle. \tag{74}$$

We limited ourselves again to identical pions (see discussion at the end of Section V). We have now the freedom to play with the phases and magnitudes of the f_i. As a matter of fact, the parameter that is of importance is

$$\xi = \frac{f_0^2}{f_+f_-}. \tag{75}$$

By choosing $\xi = 2$, we reach the situation of the maximal isospin state. The discussion in Section V shows us that the minimal values are reached for negative values of ξ. That, indeed, is also the case here. Before turning to the numerical evaluation, we would like still to point out that suitable choices of ξ can eliminate a particular isospin altogether from any combination of identical pions.

The reason for this result is that Eq. (62) implies

$$\langle I = n, I_z = 0, n \mid f \rangle = B^{-1/2} \langle 0 \mid f \rangle (\sqrt{2} f_0)^n \sum_p \frac{n!}{p!\, p!(n-2p)!} (2\xi)^{-p}. \quad (76)$$

Hence a suitable choice of ξ leads to $\langle I = n, I_z = 0, n \mid f \rangle = 0$. Once this is achieved, it follows from Eq. (65) that all $\langle I, I_z = 0, n \mid f \rangle = 0$. Thus the choice $\xi = -1$ eliminates $I = 2$, and the choice $\xi = -3$ eliminates $I = 3$.

Let us now turn to the question of the minimal isospin content. For simplicity, we choose $f_+ f_-$ as real and denote it by $x = f_+ f_-$, in agreement with the notation in Eq. (41). We find then the following results

$$\langle \mathbf{I}^2 \rangle = \frac{4xJ_1(2ix)}{iJ_0(2ix)} (1 + |f_0|^2) - 2x^2 \frac{J_2(2ix)}{J_0(2ix)} - 2x^2 + 2|f_0|^2 + 4 \operatorname{Re}(f_0^2 x),$$
$$\langle n \rangle = \langle f \mid N \mid f \rangle = \frac{2xJ_1(2ix)}{iJ_0(2ix)} + |f_0|^2 = \langle n_{ch} \rangle + \langle n_{\pi^0} \rangle. \quad (77)$$

The results for $\langle I \rangle$ vs. $\langle n \rangle$, where $\langle I \rangle \langle I + 1 \rangle = \langle \mathbf{I}^2 \rangle$, are plotted in Fig. 1 for several values of the parameter ξ. We see that for negative values of ξ they all lie very close to each other obeying

$$\langle \mathbf{I}^2 \rangle \approx \langle n \rangle. \quad (78)$$

Thus by going from the state $| F \rangle$ to $| f, Q = 0 \rangle$ we gained a factor of two in the

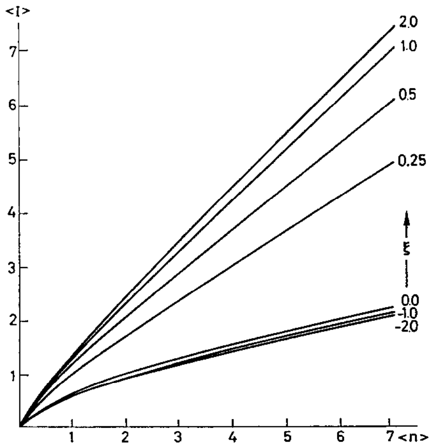

FIG. 1. Plots of $\langle I \rangle$ vs. $\langle n \rangle$ for various choices of the parameter ξ in the coherent state $| f, Q = 0 \rangle$.

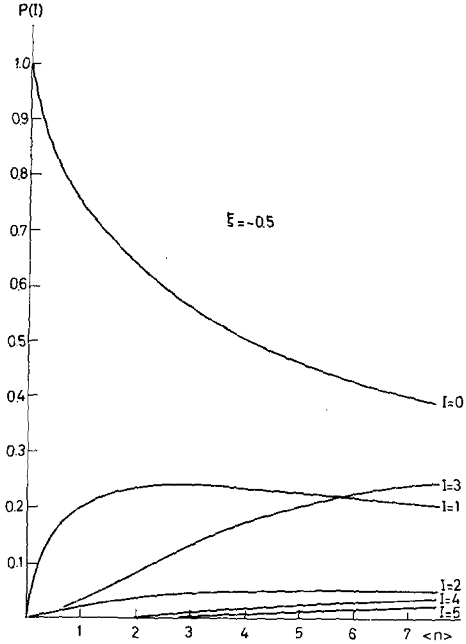

FIG. 2. The percentages of the contents of different isospins in the coherent state $|f, Q = 0\rangle$ vs. $\langle n \rangle$ for the choice $\xi = -0.5$ are plotted.

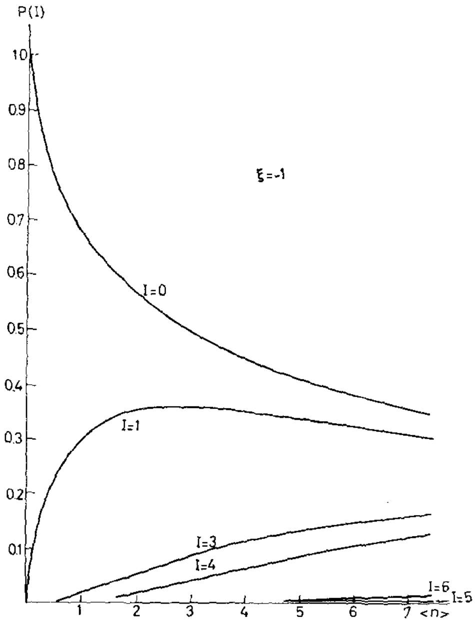

FIG. 3. Same plot as in Fig. 2 for $\xi = -1$.

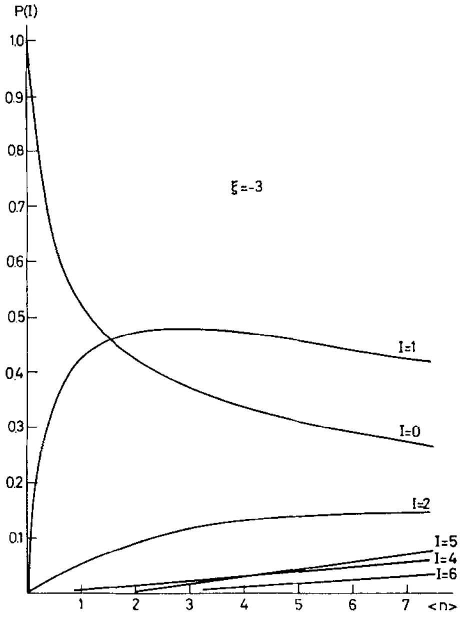

FIG. 4. Same plot as in Fig. 2 for $\xi = -3$.

minimal value of $\langle \mathbf{I}^2 \rangle$. This is of course essential in order to be able to use the coherent state $|f, Q = 0\rangle$ in a physical model.

The absolute value of ξ is the asymptotic (i.e., for large $\langle n \rangle$) ratio of the number of π^0 to the number of π^+. Therefore, we do not consider values that are too far from unity. In Figs. 2–4, we show the distribution of $\langle I \rangle$ for various choices of the parameter ξ. Figure 2 shows that for $\xi = -0.5$ all isospins higher than three are strongly quenched. Figure 3 has the choice $\xi = -1$ that eliminates $I = 2$ and Fig. 4 is drawn with $\xi = -3$ that eliminates $I = 3$. In all figures we see the important roles of low isospins for reasonable values of $\langle n \rangle$. A similar calculation leads to the distribution of the various proportions of specific isospin values in the n-pion configurations. Figure 5 shows these distributions for $\xi = -0.5$, a value that emphasizes the isospins 0, 1 and 3. Figure 6 shows the distribution for $\xi = -2$, where $I = 0$, and 1 are the important values. The relative amounts of the low isospins change slowly with ξ. If one tries to attach a coherent state to an elastic pp skeleton, one needs $I \leqslant 2$. We see from Fig. 6 that although the leading terms have low I values one still encounters sizable contributions from forbidden isospins. Hence this picture is not in very good agreement with isospin conservation. If the skeleton has higher isospins involved (like elastic πp or $pp \to p\Delta$), the situation is

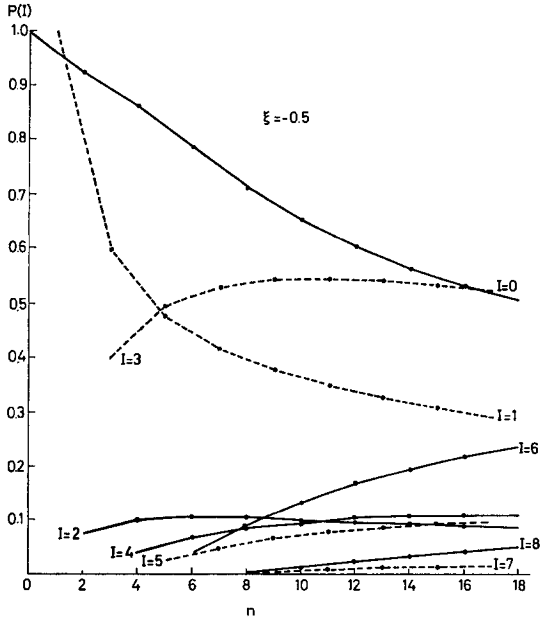

FIG. 5. The percentages of the contents of different isospins in the various n-particle states included in $| f, Q = 0\rangle$ vs. n for the choice $\xi = -0.5$ are plotted. Note the two types of curves that describe even and odd isospins for even and odd n, respectively.

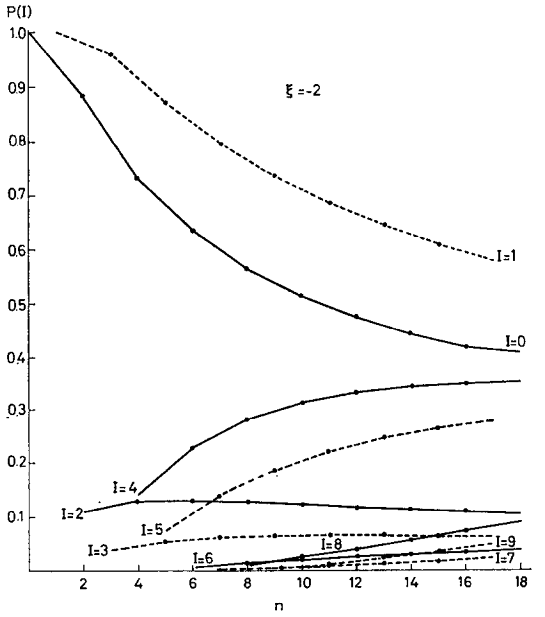

FIG. 6. Same plot as in Fig. 5 for $\xi = -2$.

better. In any case, however, the production of $|f, Q\rangle$ can only approximately satisfy the isospin restrictions.

There exists one possibility of eliminating the isospin (as well as the parity) problem altogether and that is the production of the pions in scalar isoscalar pairs. This requires of course different skeletons for even and odd pionic reactions. We will investigate here the mathematical structure of such a coherent state of identical pions. It can be called a coherent state since it can be chosen as an eigenstate of the operator A

$$A \mid g\rangle = g \mid g\rangle, \tag{79}$$

which is similar to the property (74) of $|f, Q\rangle$. The solution of Eq. (79) that is pure $I = 0$ is

$$\begin{aligned} \mid g\rangle &= \sqrt{\frac{g}{\sinh g}} \sum_m \frac{g^m}{(2m+1)!} (A^+)^m \mid 0\rangle \\ &= \sqrt{\frac{g}{\sinh g}} \sum_m \frac{g^m}{\sqrt{(2m+1)!}} \mid I = 0, n = 2m\rangle. \end{aligned} \tag{80}$$

The n-pion distribution is given by

$$P(n = 2m) = \frac{1}{\sinh g} \frac{g^{2m+1}}{(2m+1)!}. \tag{81}$$

One important property of (80) is that the isoscalar state has the same amount of all the different charges

$$\langle n_{\pi^+}\rangle = \langle n_{\pi^-}\rangle = \langle n_{\pi^0}\rangle = \tfrac{1}{3}\langle n\rangle \tag{82}$$

which is easy to check. However, now the probability of finding neutral pions is correlated to that of the charged pions (in $|f, Q\rangle$ these are independent!). The probability to find r charged pairs in a state $\mid I = 0, n = 2m\rangle$ is

$$P(r; m) = \frac{m!\, m!}{(2m+1)!} \binom{2m-2r}{m-r} 4^r. \tag{83}$$

From Eqs. (83) and (81) one can find the probability to find r charged pairs in the coherent state $\mid g\rangle$. It is

$$P_r^{(g)} = \frac{(2g)^{2r}}{\sinh g} \sum_{p=0}^{\infty} g^{2p+1} \left(\frac{(p+r)!}{p!(2p+2r+1)!}\right)^2 (2p)!\,. \tag{84}$$

The average number of pions is

$$\langle n \rangle = g \coth g - 1 \tag{85}$$

and $\langle r \rangle = \frac{1}{3}\langle n \rangle$. Using such an analysis, we can calculate the expected correlation of $\langle n_{\pi^0} \rangle$ vs. r for fixed $\langle n \rangle$. These correlations are shown in Fig. 7 where $\langle n_{\pi^0} \rangle$ is plotted versus $n_{ch} = 2 + r$ in a way that can be contrasted with the Fig. 16 in the Appendix. We see that the strong correlation that exists in Fig. 7 does not look like the trend of Fig. 16. Some other aspects of this distribution will be discussed in the next section.

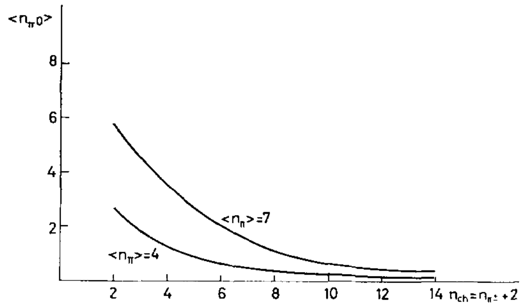

FIG. 7. The dependence of $\langle n_{\pi^0} \rangle$ on n_{π^+} is shown for two different values of $\langle n_\pi \rangle = \langle n_{\pi^0} \rangle + 2\langle n_{\pi^+} \rangle = 3\langle n_{\pi^0} \rangle$ in the coherent state $| g \rangle$.

VII. Discussion and Summary

In Section IV we saw that imposing charge conservation on a coherent state leads to obvious restrictions of the concept of independent emissions. Nevertheless, we could solve the problem within the framework of coherent states: the projection of the coherent state $| F \rangle$ of Eq. (37) on a particular charge state turned out to be the coherent state $| f, Q \rangle$ of Eq. (43). This process cannot be continued with the isospin problem. Projecting out of $| f, Q \rangle$ only the allowed isospins is possible but it distorts the nature of the state. However, we learned in Section VI that optimal choices of the parameter ξ can make these distortions minimal. Thus the coherent state $| f, Q \rangle$ can be regarded as an approximation to an allowed state. The optimal ranges of ξ depend on the skeleton to which the relevant coherent state is coupled. The higher the isospins of the skeleton particles are, the smaller the distortions of the coherent state will be. The nature of these skeletons is still left as an open problem. We may suppose that various types of skeletons can exist and their effects may interfere with each other. The resulting picture will again be very complicated unless one particular skeleton has an overwhelming effect. Under such circumstances, one can look for regularities of the cross sections for various configurations. One may for instance think of an elastic skeleton leading to a distribution $P_n^{(0)}$ of Eq. (44). In Ref. [7], we have shown that all various $P_n^{(Q)}$ look similar. We include here again the comparison of $P_n^{(0)}$ with experiment in Fig. 8. That comparison is based on the compilation of Wang [9] of many inelastic reactions. The Fig. 8 shows also distributions suggested by Wang: W^I is a Poisson

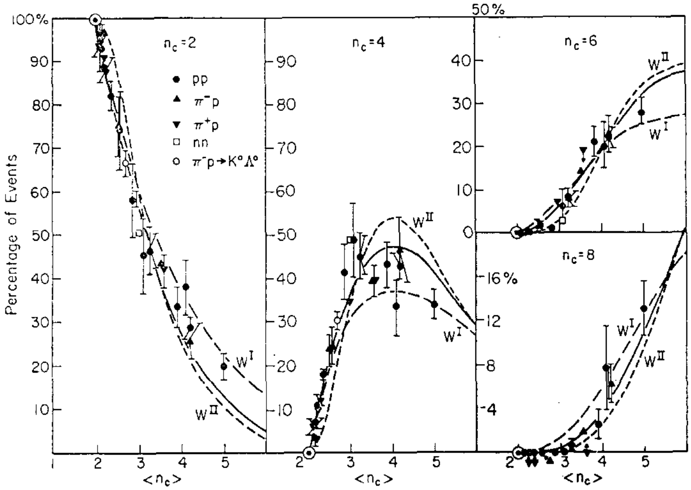

FIG. 8. Theoretical distributions are compared with Wang's compilation of inelastic production data [9]. W^I and W^{II} are taken from Wang's paper [9] and the solid curve is the result of the $P^{(0)}$ distribution following from the state $|f, Q = 0\rangle$. The figure is taken from Ref. [7].

distribution in $\frac{1}{2}(n_{ch} - 2)$ and W^{II} a hyperbolic sine distribution in $(n_{ch} - 2)$. If we replace the coherent state $|f, Q\rangle$ by the state $|g\rangle$ of Eq. (79), we find the distribution $P^{(g)}$ given by Eq. (84). Since $|g\rangle$ describes a production of pion pairs we may expect it to be somewhat similar to W^I. This is indeed the case as shown in Fig. 9, where we compare W^I, $P^{(0)}$ and $P^{(g)}$ in a fashion similar to Fig. 8. We may expect that for higher n, $P^{(0)}$ will be modified by the isospin cuts and both $P^{(0)}$ and $P^{(g)}$ will be affected by phase space limitations.

Figure 9 seems to favour the mode $|f, Q\rangle$ over $|g\rangle$. It could be expected that $P^{(g)}$ will not be a good fit since this distribution of the state $|g\rangle$ corresponds to the production of only an even number of pions. Even if another skeleton is added for the odd pions, we are still faced with the result, derived and discussed in the previous section, that a strong correlation has to be expected between $\langle n_{\pi^0}\rangle$ and n_{ch}. It is, therefore, doubtful whether this state $|g\rangle$ is actually produced in experiment. The state $|g\rangle$ has the theoretical advantage that it can be easily embedded in the formalism since its constituents are pion pairs with vacuum quantum

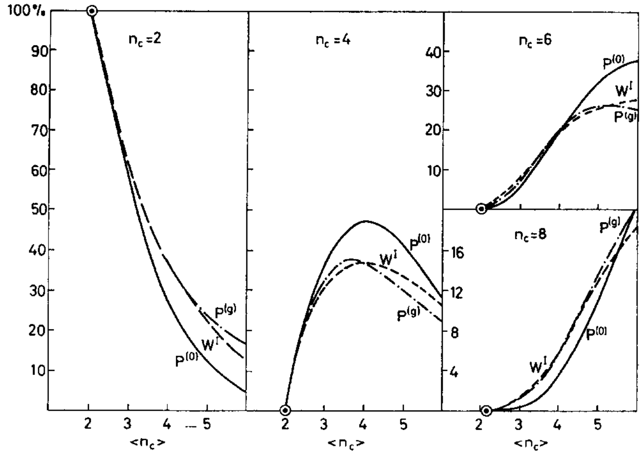

FIG. 9. Comparison of W^I, $P^{(0)}$, and $P^{(g)}$ distributions on a scale similar to Fig. 8.

numbers. Future experimental and theoretical analyses will hopefully tell us whether this mathematical simplicity can be exploited in realistic physical models.

The state $|f, Q\rangle$ is indeed closer to the concept of independent pions than the state $| g\rangle$. Throughout the paper, we have tried to follow this concept of independent production and we realize that it may look strange from a different point of view of the dynamics of multipion production. Thus the generalization of mechanisms familiar to us from two-particle productions would quite naturally lead us to expect many complicated configurations of resonances in the final states. In principle, the two different approaches may be complementary rather than contradicting. The reason is that the different production modes through resonances interfere with each other and can no longer be easily identified in the final amplitude. In such cases one usually resorts to a statistical approach. The coherent emission of pions can be regarded as a particular statistical description. We may expect, therefore, the sum of all possible production mechanisms to have a big overlap with the coherent amplitude. This leads to a new dual approach to many-particle production that should be further investigated.

Our investigation of the isospin problem answered only one necessary question: To what extent can violations of isospin inequalities be avoided. We did not give an explicit model of how the various isospins are produced. According to the picture outlined in the previous paragraph, we expect such a detailed model to be very complicated. Therefore, the correlations between various production modes of particles, that do in principle exist because of isospin conservation, do not simply follow from our description.

Our starting point was the independent production of pions and this led us to an approximate description of multipion amplitudes. There are, obviously, different statistical approaches possible. The most conventional ones use actually the conservation of isospin as their main foundation, assigning equal statistical weights to all the allowed states [10]. Their usual description does not make use of a skeleton. That is one of the differences between us and these statistical approaches. Although our model of coherent production can be regarded as a certain description of a statistical state, it does, however, incorporate dynamical assumptions and physical principles that are not of a pure statistical nature.

Finally, let us point out that our particular model for coherent production of pions — which seems to us as a natural way to approach the problem — is not necessarily the only one possible. In fact, the various papers of Ref. [1] discuss different approaches to the same problem. Our analysis in Sections IV to VI depends on the nature of the coherent state and not necessarily on the mechanism of its production (Section III). Therefore this analysis can be applied to the other theories as well. Approaching the many-pion emission problem on the basis of coherent states has some important advantages of mathematical elegance and physical simplicity, and we may hope, therefore, that it will also turn out to be useful and productive.

APPENDIX: EXPERIMENTAL EXAMPLES

In this Appendix, we discuss some experimental facts that can serve as a background for the theoretical discussion. In particular, we want to elucidate the concepts defined in the Introduction. We rely heavily on both published and

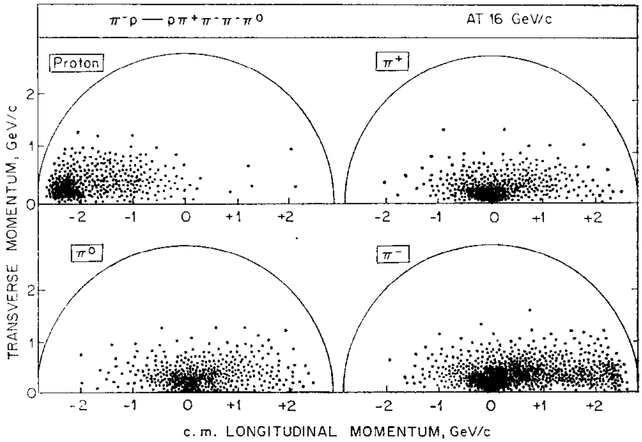

FIG. 10. Peyrou plot of $\pi^- p \to p\pi^+\pi^-\pi^-\pi^0$. Taken from Ref. [11].

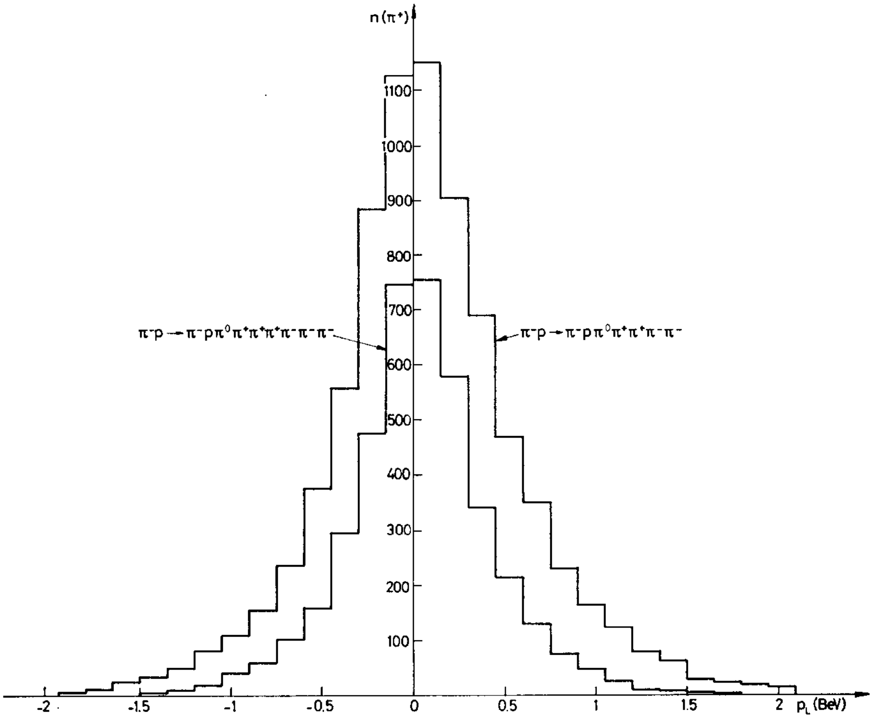

FIG. 11. Distribution of π^+ events vs. their longitudinal centre-of-mass momentum. Unpublished data of Ref. [11].

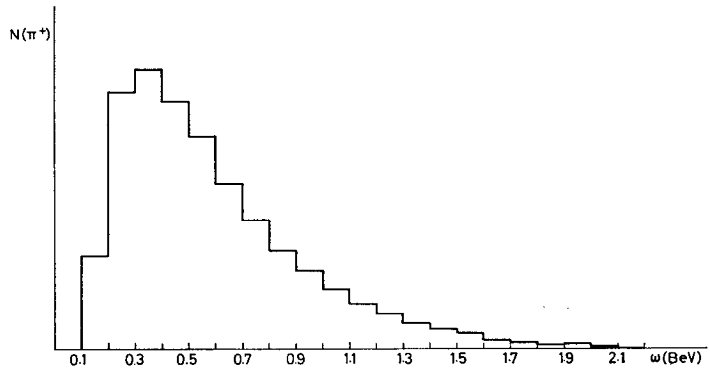

FIG. 12. Distribution of π^+ events vs. their centre-of-mass energy in the reaction $\pi^-p \to p\pi^-\pi^+\pi^+\pi^-\pi^-\pi^0$. Unpublished data of Ref. [11].

unpublished data of the ABBCCHW collaboration [11, 12] on 16 GeV $\pi^- p$ interactions.

Figure 10 shows the existence of leading particles and the concentration of the remaining pions around the centre-of-mass. This concentration is further investigated in Figs. 11 and 12. Figure 11 shows the longitudinal centre-of-mass distribution of the π^+ in two different configurations. Note the similarity between the dominant features of these two curves that would also be implied by independent production of π^+. Figure 12 gives the centre-of-mass energy distribution of the π^+ in the reaction $\pi^- p \rightarrow \pi^- p \pi^+ \pi^+ \pi^- \pi^- \pi^0$. Note the dominant peaking at energies of the order of 0.3 GeV. The strong peaking at low energies in the centre-of-mass is necessary in order for relation (29) to hold for a large range of n so that phase-space limitations do not affect the distributions characteristic of independent production.

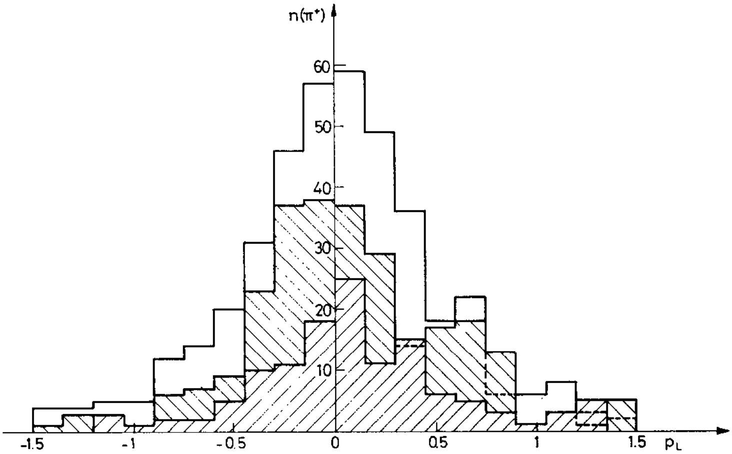

FIG. 13. Distribution of π^+ events vs. their longitudinal centre-of-mass momentum as a function of the other π^+'s longitudinal momentum in $\pi^- p \rightarrow \pi^- p \pi^+ \pi^+ \pi^- \pi^- \pi^0$. See explanation in text. Unpublished data of Ref. [11].

Let us come now to the question of uncorrelated production. Ideally, we should look at a seven-dimensional plot for two pions to check relation (1). Instead we choose to restrict ourselves to a check of the correlation between two longitudinal momentum distributions. Figure 13 shows the longitudinal centre-of-mass momentum distribution of one π^+ in $\pi^- p \rightarrow \pi^- p \pi^0 \pi^+ \pi^+ \pi^- \pi^-$ for several choices of the longitudinal momentum of the other π^+ [these choices correspond to values of (0, −0.1 GeV), (−0.2, −0.3), (−0.4, −0.5) for the three distributions shown]. The general trend is a uniform decrease of the distribution in agreement with uncorrelated production. One may argue that, since $\pi^+ \pi^+$ is an exotic channel, it should not exhibit strong correlations, whereas the $\pi^+\pi^0$ channel could be more sensitive to such effects. The truth of the matter is that a similar check for $\pi^+\pi^0$ correlations shows a far less regular behaviour than Fig. 13. Nevertheless, it is still true that the main bulk of events is concentrated around the centre of the $p_L{}^+p_L{}^0$ system. We have to conclude that uncorrelated production of $\pi^+\pi^0$ is a very crude approximation to the data.

If independent pion production can be regarded as an approximation to experiment, we may expect that the bigger the number of produced pions is the more energy is dissipated into the pionic cloud. As a consequence, the energy available for the leading particles will decrease with increasing multiplicity of pions. That this is indeed the trend of the data is seen in Fig. 14. We see that the average

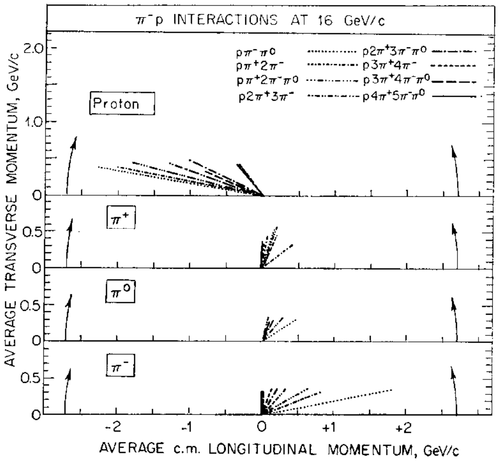

FIG. 14. Vectors of average momenta of various particles in different modes observed in the π^-p experiment. Taken from Ref. [12].

momenta of the proton and the π^- are decreasing systematically with increasing multiplicity. The other pion's behaviour does not change so drastically. Note also the almost constant value of the transverse momentum of both the leading particles and the emitted pions. This is a general property that must be common to the skeleton and the coherent state in our model.

Finally, let us discuss the cross sections distribution for the various multiplicities. Figure 15 shows the cross sections for charged nonstrange particles production in the 16 GeV π^-p experiment. The general shape is reminiscent of Poisson type distributions. For $n_{ch} = 2$, we included two entries. The dots show all inelastic processes and the cross includes also the elastic one. Using a statistical approach, one may wonder whether the elastic cross section should be included or not.

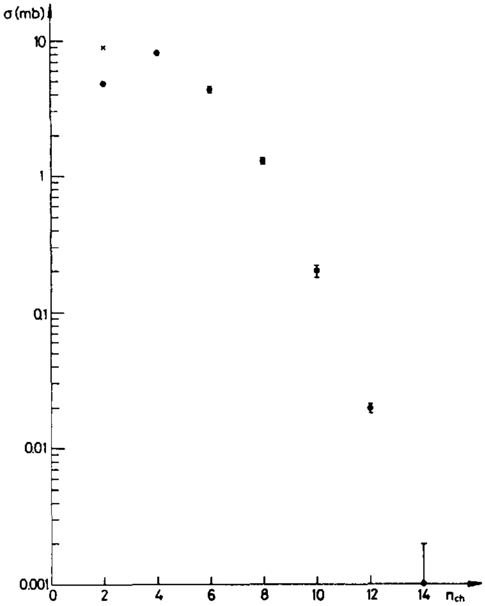

FIG. 15. Distribution of the cross section for nonstrange particle production in the $\pi^- p$ experiment. Taken from Ref. [11]. The cross represents the value obtained for $n_{ch} = 2$ if the elastic reaction is included.

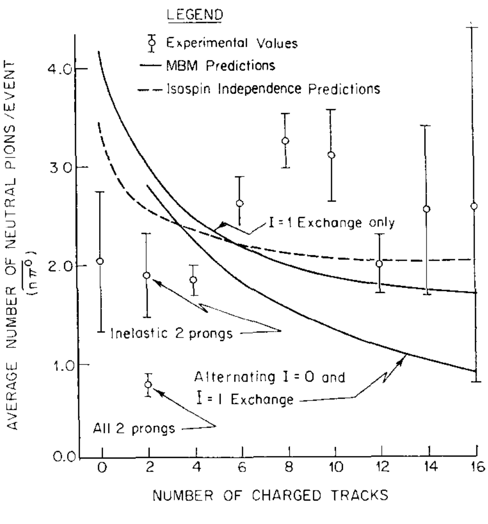

FIG. 16. The average number of π^0 produced in $\pi^- p$ at 25 GeV vs. the number of charged particles. Taken from Ref. [13].

Figure 16 (taken from a 25 GeV $\pi^- p$ experiment [13]) shows evidence against it: the elastic cross section is about equal in magnitude to all inelastic ones with $n_{ch} = 2$. In all other multiplicities, the reactions without any π^0 form a small minority of the events. Hence we may conclude that the elastic scattering has to be treated on a different footing. Note the behaviour of $\langle n_{\pi^0} \rangle$ in Fig. 16. A distribution following the state $| f, Q \rangle$ of Eq. (43) would lead to a constant $\langle n_{\pi^0} \rangle$, whereas the results of $| g \rangle$ shown in Fig. 7 do not show the trends seen in Fig. 16.

The experimental facts discussed above should be considered as a background for the ideas discussed in the paper. We are aware of the fact that different experiments show different amounts of strong resonance productions and the results shown here are not reproduced in detail in other experimental configurations. Nevertheless, the general features are similar and we may hope that they can be crudely described by the model discussed in the paper.

Acknowledgments

We would like to thank all our colleagues in Caltech for their constructive criticism and advice. We are grateful to Dr. R. Stroynowski for providing us with the unpublished data of Ref. [11] that are discussed in the Appendix. One of us (D. H.) would like to thank Professor J. Prentki for his kind hospitality at the Theory Division of CERN.

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